DOI: 10.13009/EUCASS2023-309
Aerospace Europe Conference 2023 – 10TH EUCASS – 9TH CEAS

An arbitrary order H_∞ interval filter for satellite navigation units

Irryhl Mohammedi[⋆]*, David Gucik-Derigny*[⋆]*, David Henry*[⋆]
[⋆] *University of Bordeaux, IMS-Lab, Automatic control group, 351 Cours de la liberation, 33405 Talence, France.*
irryhl.mohammedi@ims-bordeaux.fr · david.gucik-derigny@ims-bordeaux.fr · david.henry@ims-bordeaux.fr

Abstract

This paper investigates a new H_∞ interval filter design methodology and presents an innovative application of the filter developed to estimate satellite true attitude state for navigation units in space missions. The key element of the proposed approach is to use an interval filter structure advantage, rather than an observer-based structure (relying only on a system dynamics structure). Indeed, the interval filter design is an emerging research field, the design problem is formulated as an arbitrary order generic state-space realization. Thus, such state-space realization offers more degrees of freedom in its design, since it relies on a synthesis which is solved on an order which can be inferior, superior or equal to the system state space order, improving the interval state estimation performance, in a H_∞-gain criterion sense. The H_∞ theory is used to enhance robustness against sensor misalignment errors, noises unknown inputs and disturbances for satellite missions application.

1. Introduction

For spacecraft and satellite systems, state variable estimation is a key approach for controller implementation and fault detection. This issue is generally addressed using estimation algorithms integrated in the navigation module. The interval estimation method, which aims to reliably estimate the upper and lower bounds of system state, has acquired a significant attention over the last two decades. Based on monotone system theory to obtain the cooperativity property on the dynamics of observation errors;[9,14] or based on reachable set prediction/correction to build compact convex sets,[19,21] the interval-based observer or filter can be a potential solution to to estimate state for spacecraft and satellite systems control.[16] Considering the approach based on monotone system theory which is under the scope of the paper, the interval estimation method relies on a gain-performance criterion of a transfer function from unknown input, uncertainty, noise or disturbance to estimation error bounds, based on a given norm sense. The challenge is to obtain the cooperativity property of the estimator to guarantee upper and lower bounds of system state and to satisfy the performance criterion a priori by solving Linear Matrix Inequalities (LMI) to obtain thin width interval estimation. Some recent works propose interval estimator design methodologies in that context. In the work,[4] the authors consider an H_∞ criterion for interval observer design coupled with D-stability. The solution[5] proposed is formulated as a Semi-Definite Positive (SDP) problem to be solved, which allows joining L_2/L_∞ performance gain criterion for a Linear Parameter Varying (LPV) systems. A conservative aspect of the above-mentioned works is related to the observer structure and state space dimension limitation in which the estimator design problem is solved. Indeed, the design of the interval observer structure is based on the system state dynamics structure, which limits the ability to satisfy both estimator cooperativity property and the performance criterion through LMI solving. Filter design approach could be introduced to overcome such restriction. Indeed, in the literature, robust H_∞ and H_2 filter design approach has received considerable attention in estimation and control theory[2,3,7,8] for Linear Time Invariant (LTI) systems. Although, at the best of the author's knowledge, the filter design in an interval context has not been explored and remains an open problem.

In this paper, the main contributions are twofold. First, a novel approach to design a state estimation theory inspired by a state-space realization of an interval filter has been proposed. In comparison with classical filter design approaches for LTI systems, the interval filter design for uncertain LTI systems needs to solve a non-convex problem (caused by functions used to guarantee upper and lower bounds of the state), which is NP-hard to solve. The introduction of slack matrices as described in works[1,11,12] allowed to reduce the conservatism for the problem to be solved and to linearize a part of the non-convex problem. Secondly, the application of the robust interval filter technique to the guaranteed estimation of attitude angles of the satellite for an implementation into the navigation unit of the guidance, navigation, and control/attitude and orbit control systems of a satellite. The proposed approach exploits the closed-loop system

DOI: 10.13009/EUCASS2023-309

augmented by the satellite controller and the introduction of slack matrices for matrix inequalities solving purpose and approach conservatism reduction. The interval filter problem is formulated as an arbitrary order generic state-space realization. Thus, such state-space realization offers more degrees of freedom in its design, since it relies on a synthesis which is solved on a state space order which can be inferior, superior or equal to the system state space order. The H_∞ theory is used to enhance robustness against sensor misalignment errors, noises and other unknown input or disturbance experienced during satellite missions application. Here, it is shown that the design of the interval filter can be formulated as an optimization problem under linear and bilinear matrix inequalities (BMI) constraints, that can be solved using BMI solvers like PENLAB. Results obtained from the functional engineering simulator (FES) demonstrate the pertinence of the proposed approach.
This paper is structured as follows. In Section 2, some preliminaries including definitions and lemmas are provided. Section 3 discusses the satellite mission and considers modeling issues. The problem statement is presented in Section 4. The main results of designing the interval filter to a satellite mission is presented in Section 5 and 6. Section 7 is devoted to results obtained from the satellite mission simulator. Finally, some concluding remarks are reported in Section 8.

2. Preliminaries

The sets of real is denoted as $\mathbb{R}$, I_n represents the $n \times n$ dimension identity matrix and $0_{p\times m}$ denotes a $p \times m$ dimension matrix with all zero elements. The matrix transpose and the pseudo-inverse of a matrix M are denoted as M^t, $M^\dagger$, respectively. The comparison operators $\geq, >, \leq$ and $<$ on vectors and matrices are understood elementwise. For a matrix M, we denote $M^+ = max(M, 0)$ and $M^- = M^+ - M$ (the same notations are adopted for vectors). A diagonal-block matrix composed of the elements of M is denoted $diag(M)$. For a real-valued matrix $P \in \mathbb{R}^{n\times n}$, $P \succ 0$ ($P \prec 0$) indicates that P is strictly positive (negative) definite. $G_{a\to b}$ denotes the transfer function from the input a to the output b. For a linear transfer G, its H_∞ norm is denoted $||G||_\infty = \sup\limits_{\omega\in\mathbb{R}} \overline{\sigma}(G(j\omega)) \leq \infty$, where $\overline{\sigma}(.)$ refers to the maximum singular value. Given a vector x and two vectors $\underline{x}, \overline{x} \in \mathbb{R}^m$, $\left[\underline{x}, \overline{x}\right]$ denote an interval subset of $\mathbb{R}^m$, where $\underline{x}$ and $\overline{x}$ stand respectively for its lower and upper bounds such as $\underline{x} \leq x \leq \overline{x}$.

Definition 1 [17] *Let be a square matrix W whose elements satisfy $w_{ij} \leq 0, \forall i \neq j$ and $w_{ij} > 0, \forall i = j$ then W is said to be an M-matrix if W^{-1} exists and $W^{-1} \geq 0$.*

Lemma 1 [5] *Consider a matrix $A \in \mathbb{R}^{n\times n}$, an invertible M-matrix $W \in \mathbb{R}^{m\times n}$ and a real scalar $\mu \geq 0$. Then A is a Metzler matrix if $WA + \mu W \geq 0$.*

Lemma 2 [1] *Consider a transfer function G with the state space realization $(\overline{A}, \overline{B}, \overline{C}, \overline{D})$. Then $||G||_\infty \leq \gamma$, if there exist auxiliary matrices W, Z, J, M and symmetric matrix $P = P^t$ of adequate dimensions, satisfying the following LMIs*

$$P = P^t \succ 0, \tag{1}$$

$$\underbrace{\begin{bmatrix} 0 & P & 0 & 0 \\ P & 0 & 0 & \overline{C}^t \\ 0 & 0 & -I & \overline{D}^t \\ 0 & \overline{C} & \overline{D} & -\gamma I \end{bmatrix}}_{\Omega} + \underbrace{\begin{bmatrix} -W & W\overline{A} & W\overline{B} & 0 \\ -Z & Z\overline{A} & Z\overline{B} & 0 \\ -J & J\overline{A} & J\overline{B} & 0 \\ -M & M\overline{A} & M\overline{B} & 0 \end{bmatrix}}_{\Upsilon} + \Upsilon^t \prec 0. \tag{2}$$

Lemma 3 *Given matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{r\times m}$, let define some matrices $B_k \in \mathbb{R}^{r\times m}$, $k \in \{a, b\}$ such that $B = B_a - B_b$, with B_a, $B_b \geq 0$ and a vector $x \in [\underline{x}, \overline{x}] \in \mathbb{R}^n$. Since matrices A and B are constant, then*

$$(B_aA^+ + B_bA^-)\underline{x} - (B_aA^- + B_bA^+)\overline{x} \leq BAx \leq (B_aA^+ + B_bA^-)\overline{x} - (B_aA^- + B_bA^+)\underline{x} \tag{3}$$

Proof 1 *Note that*

$$BAx = (B_a - B_b)(A^+ - A^-)x = (B_aA^+ - B_aA^- - B_bA^+ + B_bA^-)x, \tag{4}$$

which for $\underline{x} \leq x \leq \overline{x}$ gives

$$B_aA^+\underline{x} - B_aA^-\overline{x} - B_bA^+\overline{x} + B_bA^-\underline{x} \leq BAx \leq B_aA^+\overline{x} - B_aA^-\underline{x} - B_bA^+\underline{x} + B_bA^-\overline{x} \tag{5}$$

equivalent to the required inequalities (3).

DOI: 10.13009/EUCASS2023-309

3. The satellite mission

For reasons of brevity, we will attempt to avoid repeating the technical background presented in the work.[10] To this end, the focus of this section will lie wholly with the model given by Eqs. (6a)-(6b) and we invite the reader to refer to previous mentioned work[10] for further details.
The application support is inspired by the Microscope satellite mission.[13] The satellite combines simultaneously a rotational motion ω_o around the Earth on a sun-synchronous, quasi-circular dawn-dusk orbit, while performing simultaneously a rotation around its y-axis ω_s, see Figure 1 and the works.[10,13,18] The tracking of the trajectory is ensured by a AACS (Attitude and Acceleration Control System), whose schematic structure is illustrated in Figure 1. The references are set to '0', which means that the satellite is enforced to ensure its rotation around Earth at velocity ω_o while spinning at ω_s around its y-axis. The avionics is assumed to be composed of a μASC (micro Advanced Stellar Compass) and the SAGE (Space Accelerometer for Gravitation Experimentation) system, that provide noisy measurements of the attitude $\Theta = [\phi\,\theta\,\psi]^t \in \mathbb{R}^3$ and both the angular accelerations $\dot{\omega} = [\dot{p}\,\dot{q}\,\dot{r}]^t \in \mathbb{R}^3$ and linear acceleration $\Gamma = [\Gamma_x\,\Gamma_y\,\Gamma_z]^t \in \mathbb{R}^3$. We also denote Θ_m, $\dot{\omega}_m$ and Γ_m respectively with the index "m" associated for the measured variables. The navigation unit is in charge of providing an estimate of the linear acceleration Γ (role of the hybridation filter), the lower and upper bounds $\underline{\Theta}, \overline{\Theta} \in \mathbb{R}^3$ of the true attitude Θ (role of the H_∞ interval filter), and an estimate $\hat{\Theta}$ of the attitude, derived by means of a l_1-optimal fusion rule (role of the fusion algorithm). The control is ensured by a 6DOF controller that delivers the 3-dimension force $F_c \in \mathbb{R}^3$ and torque $C_c \in \mathbb{R}^3$, that are then converted to thrusters firing $T_C \in \mathbb{R}^{12}$ by means of the thrusters management unit.

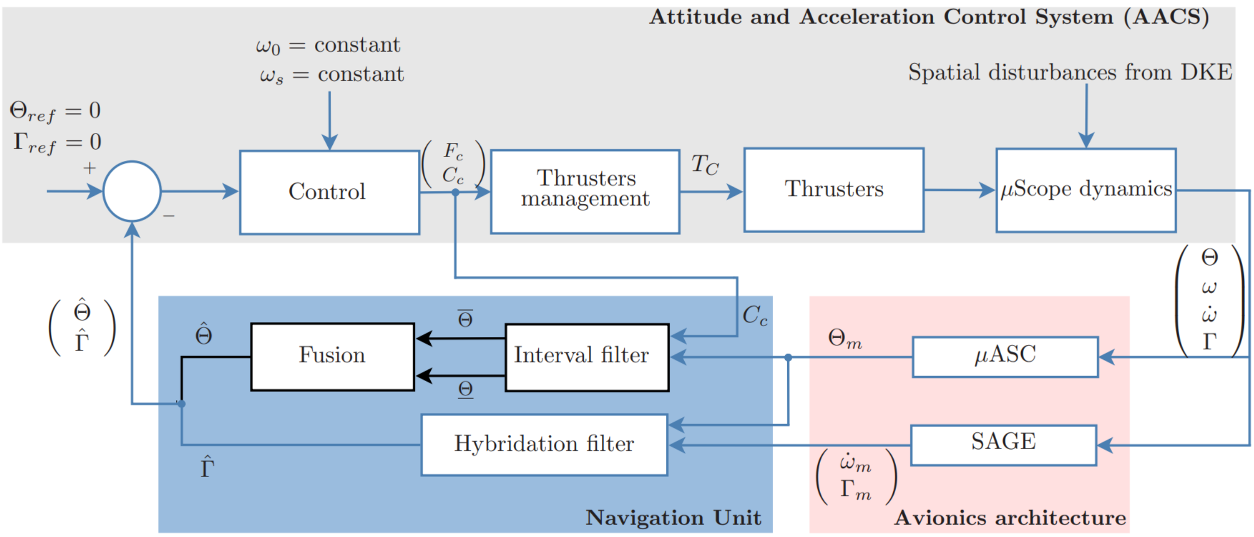

Figure 1: AACS and avionics architecture

The satellite mission is implemented into a functional engineering simulator (FES), which includes highly representative models of sensors (misalignment errors, non Gaussian noises...) and actuators (including dead-zone, Minimum Impulse Bit effect...), and Dynamics Kinematics and Environment models. The environment modules contain the spatial disturbances such as the magnetic field, the aerodynamic drag, the gravitational disturbances, the solar and the albedo radiations. The interested reader can refer to the work[10] to get a complete description of all models described in Figure 1.

4. Problem statement

The satellite attitude dynamics can be modelled as the following linear time invariant model:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ew(t) & \text{(6a)} \\ y(t) = Cx(t) + Fw(t) & \text{(6b)} \\ z(t) = Tx(t) & \text{(6c)} \end{cases}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times n_u}, E \in \mathbb{R}^{n\times n_w}$, $C \in \mathbb{R}^{n_y\times n}$, $F \in \mathbb{R}^{n_y\times n_w}$ and $T \in \mathbb{R}^{n_z\times n}$, with $n = 6$, $n_u = 3$ and $n_y = 3$. $x = (\phi\ \theta\ \psi\ p\ q\ r)^t$, $u = C_c$, $y = \Theta_m$ and $z = (\phi\ \theta\ \psi)^t$ are the state vector, the torque command delivered by the control

DOI: 10.13009/EUCASS2023-309

unit, the attitude angles delivered by the μASC sensor, and the signals to be estimated, respectively. $w \in \mathbb{R}^{n_w}$ with $n_w = 6$, refer to the spatial disturbances given in torques, and the sensor misalignment errors and noises. The initial condition $x(0)$ and the unknown input $w(t)$ are bounded with a priori known bounds i.e. for $\underline{w}(t), \overline{w}(t) \in \mathbb{R}^{n_w}$ and $\underline{x}(0), \overline{x}(0) \in \mathbb{R}^n$, we have $w(t) \in \left[\underline{w}(t), \overline{w}(t)\right]$, $\forall t \in \mathbb{R}_+$ and $x(0) \in \left[\underline{x}(0), \overline{x}(0)\right]$.

The 3DOF controller that delivers the three-dimensional torque C_c, admits the following state space representation

$$\begin{cases} \dot{x}_k(t) = A_K x_k(t) - B_K y(t) & \text{(7a)} \\ u(t) = C_K x_k(t) - D_K y(t) & \text{(7b)} \end{cases}$$

where $x_k \in \mathbb{R}^{n_k}$.
Therefore, the closed-loop system augmented by the controller is governed by

$$\begin{cases} \dot{x}(t) = (A - BD_K C)x(t) + BC_K x_k(t) + (E - BD_K F)w(t) & \text{(8a)} \\ \dot{x}_k(t) = A_K x_k(t) - B_K C x(t) - B_K F w(t) & \text{(8b)} \\ y(t) = Cx(t) + Fw(t) & \text{(8c)} \\ z(t) = Tx(t) & \text{(8d)} \end{cases}$$

Let now consider the interval filter for the system (8a)-(8d)

$$\begin{cases} \dot{\overline{s}}_f(t) = A_F \overline{s}_f(t) + (B_{F_1} - B_F)y(t) + \overline{\epsilon}_1(\overline{z}, \underline{z}, \overline{w}, \underline{w}) & \text{(9a)} \\ \dot{\underline{s}}_f(t) = A_F \underline{s}_f(t) + (B_F - B_{F_2})y(t) + \underline{\epsilon}_2(\overline{z}, \underline{z}, \overline{w}, \underline{w}) & \text{(9b)} \\ \dot{x}_f(t) = A_F x_f(t) + M_F u(t) + B_F y(t) & \text{(9c)} \\ \overline{x}_f(t) = \overline{s}_f(t) + x_f(t) & \text{(9d)} \\ \underline{x}_f(t) = x_f(t) - \underline{s}_f(t) & \text{(9e)} \\ \overline{z}(t) = C_{F_1}\overline{x}_f(t) + D_F y(t) + \overline{\eta}_1(\overline{x}_f, \underline{x}_f, \overline{w}, \underline{w}) & \text{(9f)} \\ \underline{z}(t) = C_{F_2}\underline{x}_f(t) + D_F y(t) + \underline{\eta}_2(\overline{x}_f, \underline{x}_f, \overline{w}, \underline{w}) & \text{(9g)} \end{cases}$$

Here, the dynamics of the filter states $\overline{s}_f(t), \underline{s}_f(t) \in \mathbb{R}^{n_f}$ are supposed to satisfy the cooperativity property, i.e $\overline{s}_f(t), \underline{s}_f(t) \geq 0, \forall t \geq 0$, under initial conditions given by $\underline{s}_f(0), \overline{s}_f(0) \geq 0$. Then $\overline{x}_f(t)$, $\underline{x}_f(t) \in \mathbb{R}^{n_f}$ denote the upper and lower bounds of the state $x_f(t)$ satisfying the inequality $\underline{x}_f(t) \leq x_f(t) \leq \overline{x}_f(t)$, $\forall t \geq 0$, under initial conditions given by $\underline{x}_f(0) \leq x_f(0) \leq \overline{x}_f(0)$. Vectors $\overline{z}(t)$, $\underline{z}(t) \in \mathbb{R}^{n_z}$ are the filter outputs that theoretically guarantee that the true vector $\Theta(t)$ belongs to the interval $[\underline{z}(t), \overline{z}(t)]$. The objective is to determine the matrices $A_F, B_F, M_F, B_{F_i}, C_{F_i}$ and D_F of adequate dimensions, with $i \in \{1, 2\}$, and the functions $\overline{\epsilon}_1, \underline{\epsilon}_2 : \mathbb{R}^{n_z} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_f}$ and $\overline{\eta}_1, \underline{\eta}_2 : \mathbb{R}^{n_f} \times \mathbb{R}^{n_f} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_z}$, so that the propagation effect of the vectors $w(t), \overline{w}(t)$ and $\underline{w}(t)$ on the estimation error length defined as $\tilde{e}_z(t) = \overline{z}(t) - \underline{z}(t)$ is minimized, in the H_∞ norm criterion sense.

Assumption 1 *The dimension of the interval filter state, here $\overline{s}_f(t)$ and $\underline{s}_f(t) \in \mathbb{R}^{n_f}$, is arbitrary such as $n_f \neq n$ or $n_f = n$.*

Assumption 2 *The matrix $T \in \mathbb{R}^{n_z \times n}$, with the dimension $n_z \leq n$, has linearly independent rows such that right pseudo-inverse of matrix T exists and satisfies $D_F C T^\dagger = I_{n_z}$.*

Remark 1 *In the case where $n_z = n$, the matrix T is supposed to be invertible and satisfies $D_F C T^{-1} = I_{n_z}$.*

5. Design of an H_∞ interval filter

5.1 Necessary and sufficient conditions

The following theorem gives the existence conditions for interval filter (9a)-(9g). Hereafter, the time dependence of signals is omitted to improve the presentation clarity, when it is possible.

DOI: 10.13009/EUCASS2023-309

Theorem 1 *Under assumptions 1-2, the state $z(t)$ of the stable system (8a)-(8d) is bounded by $\overline{z}, \underline{z}$ defined in Eqs.* (9f)-(9g) *of interval filter* (9a)-(9g) *such that*

$$\underline{z}(t) \leq z(t) \leq \overline{z}(t), \quad \forall t \geq 0 \tag{10}$$

iff the following conditions hold

1. *There exists a matrix A_F such that*
$$A_F \text{ is Metzler}. \tag{11}$$

2. *There exists some matrices C_{F_i}, $i \in \{1, 2\}$ and D_F such that*
$$C_{F_i} \geq 0, \tag{12a}$$
$$D_F C T^\dagger = I_{n_z}. \tag{12b}$$

3. *Some functions $\overline{\epsilon}_1, \underline{\epsilon}_2, \overline{\eta}_1, \underline{\eta}_2$ exist satisfying*
$$\overline{\epsilon}_1(\overline{z}, \underline{z}, \overline{w}, \underline{w}) - \epsilon_1(z, w) \geq 0, \tag{13a}$$
$$\epsilon_2(z, w) - \underline{\epsilon}_2(\overline{z}, \underline{z}, \overline{w}, \underline{w}) \geq 0, \tag{13b}$$
$$\overline{\eta}_1(\overline{x}_f, \underline{x}_f, \overline{w}, \underline{w}) - \eta_1(x_f, w) \geq 0, \tag{13c}$$
$$\eta_2(x_f, w) - \underline{\eta}_2(\overline{x}_f, \underline{x}_f, \overline{w}, \underline{w}) \geq 0, \tag{13d}$$
where
$$\epsilon_i(z, w) = R_i Y z + R_i S w, \tag{14a}$$
$$\eta_i(x_f, w) = C_{F_i}(-I_{n_f})x_f + D_F(-F)w, \tag{14b}$$
with $R_i = \begin{bmatrix} B_{F_i} & B_F \end{bmatrix}$, $Y = \begin{bmatrix} -CT^\dagger & CT^\dagger \end{bmatrix}^t$, $S = \begin{bmatrix} -F & F \end{bmatrix}^t$ *and* $i \in \{1, 2\}$.

4. *The initial conditions for $\overline{s}_f$, $\underline{s}_f$, $\overline{x}_f$, $\underline{x}_f$ and x_f are given by*
$$\underline{s}_f(0), \overline{s}_f(0) \geq 0, \tag{15}$$
$$\underline{x}_f(0) \leq x_f(0) \leq \overline{x}_f(0), \tag{16}$$
such that
$$\underline{s}_f(t), \overline{s}_f(t) \geq 0, \tag{17}$$
$$\underline{x}_f(t) \leq x_f(t) \leq \overline{x}_f(t), \ \forall t \geq 0. \tag{18}$$

Proof 2 *Consider the relation $x = T^\dagger z$ and the functions $\epsilon_i(z, w)$, $\eta_i(x_f, w)$ in Eqs. (14a)-(14b). The dynamics of states $\overline{s}_f(t)$ and $\underline{s}_f(t)$ are given by*

$$\dot{\overline{s}}_f(t) = A_F \overline{s}_f(t) + \overline{\epsilon}_1(\overline{z}, \underline{z}, \overline{w}, \underline{w}) - \epsilon_1(z, w), \tag{19a}$$
$$\dot{\underline{s}}_f(t) = A_F \underline{s}_f(t) + \epsilon_2(z, w) - \underline{\epsilon}_2(\overline{z}, \underline{z}, \overline{w}, \underline{w}). \tag{19b}$$

Under the Metzler property (11)*, the condition* (13a)-(13b)*, and the initial conditions (15), it can be ensured that (19a)-(19b) satisfies the cooperativity property and consequently*

$$\overline{s}_f(t), \underline{s}_f(t) \geq 0, \forall t \geq 0. \tag{20}$$

From the Eqs. (9d)-(9e), the initial conditions (16), and considering that the cooperativity property (20) is satisfied, it can be directly deduced that $\overline{x}_f(t)$ and $\underline{x}_f(t)$ satisfy

$$\underline{x}_f(t) \leq x_f(t) \leq \overline{x}_f(t), \ \forall t \geq 0. \tag{21}$$

DOI: 10.13009/EUCASS2023-309

By combining $x(t) = T^{\dagger}z(t)$, (8c) *and (9d)-(9e), it can be inferred that*

$$\overline{z} = C_{F_1}\overline{s}_f + C_{F_1}x_f + D_F C T^{\dagger} z + D_F F w + \overline{\eta}_1(\cdot) \tag{22a}$$
$$\underline{z} = C_{F_2}x_f - C_{F_2}\underline{s}_f + D_F C T^{\dagger} z + D_F F w + \underline{\eta}_2(\cdot). \tag{22b}$$

Under the constraint of equality (12b)*, then*

$$\overline{z} - z = C_{F_1}\overline{s}_f + \overline{\eta}_1(\cdot) - (-C_{F_1}x_f - D_F F w) \tag{23a}$$
$$z - \underline{z} = C_{F_2}\underline{s}_f + (-C_{F_2}x_f - D_F F w) - \underline{\eta}_2(\cdot). \tag{23b}$$

Finally, under the matrix constraints (12a)*, the conditions* (13c)-(13d)*,* (14b)*, and if (17) and (18) are satisfied, the existence of an interval filter for the system (8a)-(8d) is fulfilled. This implies that the bounds* $\overline{z}, \underline{z}$ *defined in* (9f)-(9g) *satisfy the inequality* $\underline{z}(t) \leq z(t) \leq \overline{z}(t)$, $\forall t \geq 0$, *concluding the proof.*

5.2 Determination of functions $\overline{\epsilon}_1, \underline{\epsilon}_2, \overline{\eta}_1$ and $\underline{\eta}_2$

The upper and lower bounds for functions (14a) and (14b), satisfying (13a)-(13d) are now addressed. Let each unknown filter matrix B_F, B_{F_i}, C_{F_i} and D_F, $i \in \{1, 2\}$ in (9a)-(9g) be decomposed into two positive (non unique) matrices such that

$$K_{F_i} = K_{F_{ia}} - K_{F_{ib}}, \tag{24a}$$
$$X_F = X_{F_a} - X_{F_b}, \tag{24b}$$

where $K_{F_{ia}}, K_{F_{ib}} \geq 0$ with $K_{F_i} \in \{B_{F_i}, C_{F_i}\}$, and $X_{F_a}, X_{F_b} \geq 0$ with $X_F \in \{B_F, D_F\}$. Then the following decomposition for the matrix R_i for (14a) is also considered

$$R_i = R_{ia} - R_{ib}, \tag{25}$$

where $R_{ia} = \begin{bmatrix} B_{F_{ia}} & B_{Fa} \end{bmatrix}$ and $R_{ib} = \begin{bmatrix} B_{F_{ib}} & B_{Fb} \end{bmatrix}$, such that $B_{F_{ia}}, B_{F_{ib}}, B_{Fa}, B_{Fb} \geq 0$.

On the obtained expression, the functions $\overline{\epsilon}_1, \underline{\epsilon}_2 : \mathbb{R}^{n_z} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_f}$ and $\overline{\eta}_1, \underline{\eta}_2 : \mathbb{R}^{n_f} \times \mathbb{R}^{n_f} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_z}$ are given by the following Lemma.

Lemma 4 *Let consider decompositions (24a)-(25) for (14a) and (24a)-(24b) for (14b). Then, for* $x_f \in \left[\overline{x}_f, \underline{x}_f\right]$ *and* $z \in \left[\overline{z}, \underline{z}\right]$*, the inequalities* (13a)-(13d) *of Theorem 1 holds with*

$$\overline{\epsilon}_1(\overline{z}, \underline{z}, \overline{w}, \underline{w}) = \Sigma_{11}\overline{z} - \Sigma_{12}\underline{z} + \Psi_{11}\overline{w} - \Psi_{12}\underline{w}, \tag{26a}$$
$$\underline{\epsilon}_2(\overline{z}, \underline{z}, \overline{w}, \underline{w}) = \Sigma_{21}\underline{z} - \Sigma_{22}\overline{z} + \Psi_{21}\underline{w} - \Psi_{22}\overline{w}, \tag{26b}$$
$$\overline{\eta}_1(\overline{x}_f, \underline{x}_f, \overline{w}, \underline{w}) = \Xi_{11}\overline{x}_f - \Xi_{12}\underline{x}_f + \Pi_{11}\overline{w} - \Pi_{12}\underline{w}, \tag{26c}$$
$$\underline{\eta}_2(\overline{x}_f, \underline{x}_f, \overline{w}, \underline{w}) = \Xi_{21}\underline{x}_f - \Xi_{22}\overline{x}_f + \Pi_{11}\underline{w} - \Pi_{12}\overline{w}, \tag{26d}$$

where

$$\begin{aligned} \Sigma_{i1} &= R_{ia}Y^{+} + R_{ib}Y^{-}, & \Xi_{i1} &= C_{Fib}(-I_{n_f})^{-}, \\ \Sigma_{i2} &= R_{ia}Y^{-} + R_{ib}Y^{+}, & \Xi_{i2} &= C_{Fia}(-I_{n_f})^{-}, \\ \Psi_{i1} &= R_{ia}S^{+} + R_{ib}S^{-}, & \Pi_{11} &= D_{Fa}(-F)^{+} + D_{Fb}(-F)^{-}, \\ \Psi_{i2} &= R_{ia}S^{-} + R_{ib}S^{+}, & \Pi_{12} &= D_{Fa}(-F)^{-} + D_{Fb}(-F)^{+}, \end{aligned}$$

with $i \in \{1, 2\}$.

Proof 3 *A direct application of Lemma 3 and decompositions (24a) - (25) on the different parts of the equations* (14a)-(14b)*, leads to upper bounds* $\overline{\epsilon}_1, \overline{\eta}_1$ *of the elements* ϵ_1, η_1 *and the lower bounds* $\underline{\epsilon}_2, \underline{\eta}_2$ *of the elements* ϵ_2, η_2 *defined in Eqs.* (26a)-(26d).

DOI: 10.13009/EUCASS2023-309

6. Synthesis of the H_∞ interval filter

This section presents the SDP formulation based on the H_∞-norm criterion to design the interval filter for its application to the problem of interval estimation of satellite attitude angles.

Based on Lemma 4, the interval filter introduced in Eqs (9a)-(9g) can be rewritten as follows

$$\dot{\overline{s}}_f(t) = A_F\overline{s}_f(t) + (B_{F_1} - B_F)y(t) + \Sigma_{11}\overline{z}_f - \Sigma_{12}\underline{z}_f + \Psi_{11}\overline{w} - \Psi_{12}\underline{w}, \tag{27}$$

$$\dot{\underline{s}}_f(t) = A_F\underline{s}_f(t) + (B_F - B_{F_2})y(t) + \Sigma_{21}\underline{z}_f - \Sigma_{22}\overline{z}_f + \Psi_{21}\underline{w} - \Psi_{22}\overline{w}, \tag{28}$$

$$\dot{x}_f(t) = A_F x_f(t) + M_F u(t) + B_F y(t), \tag{29}$$

$$\overline{z}_f(t) = C_{F_1}(\overline{s}_f(t) + x_f(t)) + D_F y(t) + \Xi_{11}\overline{x}_f - \Xi_{12}\underline{x}_f + \Pi_{11}\overline{w} - \Pi_{12}\underline{w}, \tag{30}$$

$$\underline{z}_f(t) = C_{F_2}(x_f(t) - \underline{s}_f(t)) + D_F y(t) + \Xi_{21}\underline{x}_f - \Xi_{22}\overline{x}_f + \Pi_{11}\underline{w} - \Pi_{12}\overline{w}. \tag{31}$$

Now, by replacing the expressions of $u(t), y(t), \overline{x}_f(t), \underline{x}_f(t), \overline{z}_f(t), \underline{z}_f(t)$ from (7b), (8c),(9d)-(9g) in (27)-(29), and considering the augmented state and input vector $\hat{x} = \begin{pmatrix} x^t & x_k^t & x_f^t & \overline{s}_f{}^t & \underline{s}_f{}^t \end{pmatrix}^t$, $\omega = \begin{pmatrix} w^t & \overline{w}^t & \underline{w}^t \end{pmatrix}^t$ respectively, the following extended state model is proposed

$$\begin{pmatrix} \dot{\hat{x}} \\ \tilde{e}_z \end{pmatrix} = \left[\begin{array}{c|c} \overline{A} & \overline{B} \\ \hline \overline{C} & \overline{D} \end{array}\right] \begin{pmatrix} \hat{x} \\ \omega \end{pmatrix}, \tag{32}$$

where $\overline{A}, \overline{B}, \overline{C}$ and $\overline{D}$ are given by

$$\overline{A} = \begin{bmatrix} (A - BD_KC) & BC_K & 0 \\ -B_KC & A_K & 0 \\ (B_F - M_FD_K)C & M_FC_K & A_F \\ (B_{F_1} - B_F + \Sigma_{11}D_F - \Sigma_{12}D_F)C & 0 & \Sigma_{11}(C_{F_1} + \Xi_{11} - \Xi_{12}) - \Sigma_{12}(C_{F_2} + \Xi_{21} - \Xi_{22}) \\ (B_F - B_{F_2} + \Sigma_{21}D_F - \Sigma_{22}D_F)C & 0 & \Sigma_{21}(C_{F_2} + \Xi_{21} - \Xi_{22}) - \Sigma_{22}(C_{F_1} + \Xi_{11} - \Xi_{12}) \end{bmatrix} \cdots$$

$$\cdots \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ A_F + \Sigma_{11}(C_{F_1} + \Xi_{11}) + \Sigma_{12}\Xi_{22} & \Sigma_{11}\Xi_{12} + \Sigma_{12}(C_{F_2} + \Xi_{21}) \\ -(\Sigma_{21}\Xi_{22} + \Sigma_{22}C_{F_1} + \Sigma_{22}\Xi_{11}) & A_F - \Sigma_{21}C_{F_2} - \Sigma_{21}\Xi_{21} - \Sigma_{22}\Xi_{12} \end{bmatrix}, \tag{33}$$

$$\overline{B} = \begin{bmatrix} (E - BD_KF) & 0 & 0 \\ -B_KF & 0 & 0 \\ (B_F - M_FD_K)F & 0 & 0 \\ (B_{F_1} - B_F + \Sigma_{11}D_F - \Sigma_{12}D_F)F & (\Sigma_{11}\Pi_{11} + \Sigma_{12}\Pi_{12} + \Psi_{11}) & -(\Sigma_{11}\Pi_{12} + \Sigma_{12}\Pi_{11} + \Psi_{12}) \\ (B_F - B_{F_2} + \Sigma_{21}D_F - \Sigma_{22}D_F)F & -(\Sigma_{21}\Pi_{12} + \Sigma_{22}\Pi_{11} + \Psi_{22}) & (\Sigma_{21}\Pi_{11} + \Sigma_{22}\Pi_{12} + \Psi_{21}) \end{bmatrix}, \tag{34}$$

$$\overline{C} = \begin{bmatrix} 0 & 0 & (C_{F_1} - \Xi_{12} + \Xi_{11} - C_{F_2} - \Xi_{21} + \Xi_{22}) & (C_{F_1} + \Xi_{11} + \Xi_{22}) & (\Xi_{12} + \Xi_{21} + C_{F_2}) \end{bmatrix}, \tag{35}$$

$$\overline{D} = \begin{bmatrix} 0 & (\Pi_{11} + \Pi_{12}) & -(\Pi_{12} + \Pi_{11}) \end{bmatrix}. \tag{36}$$

Here, the objective of the contribution is to minimize the influence of input ω on interval width $\tilde{e}_z$ in the sense of a H_∞-norm criterion and in such a way that filter error state space (32) is asymptotically stable. Lemma 2 fulfills this specification, however, due to the multiplicative relationships between the slack matrices and the filter matrices, this lemma is not directly applicable to the filter design. The filter design is therefore a non-convex problem of infinite dimensions, i.e NP-hard problem to solve. The Lyapunov function $V(\hat{x}) = \hat{x}^t P\hat{x}$ guarantees the asymptotic stability of the system (32). As outlined in the work,[1] the Lyapunov matrix P has no multiplication relation with the state space matrices (32) and auxiliary matrices, so it can be easily considered as a decision variable. However, by imposing a special structure on the auxiliary variables, we can formulate the design problem in an LMI and BMI to solve. Let consider slacks matrices $W, Z \in \mathbb{R}^{(n+n_k+3n_f)\times(n+n_k+3n_f)}$, $J \in \mathbb{R}^{3n_w\times(n+n_k+3n_f)}$, $M \in \mathbb{R}^{n_z\times(n+n_k+3n_f)}$ be partitioned with the following structure

DOI: 10.13009/EUCASS2023-309

$$W = diag\{W_1, W_2, W_3, W_3, W_3\}, \tag{37}$$
$$Z = diag\{Z_1, Z_2, W_3, W_3, W_3\}, \tag{38}$$
$$J = \begin{bmatrix} J_1 & J_2 & 0 & 0 & 0 \end{bmatrix}, \tag{39}$$
$$M = \begin{bmatrix} M_1 & M_2 & 0 & 0 & 0 \end{bmatrix}, \tag{40}$$

where $W_1, Z_1 \in \mathbb{R}^{n\times n}$, $W_2, Z_2 \in \mathbb{R}^{n_k\times n_k}$, $J_1 \in \mathbb{R}^{3n_w\times n}$, $J_2 \in \mathbb{R}^{3n_w\times n_k}$, $M_1 \in \mathbb{R}^{n_z\times n}$, $M_2 \in \mathbb{R}^{n_z\times n_k}$, $W_3 \in \mathbb{R}^{n_f\times n_f}$ and the following change of variables

$$\begin{bmatrix} \tilde{B}_{F_{1j}} & \tilde{B}_{F_{2j}} \\ \tilde{C}_{F_{1j}} & \tilde{C}_{F_{2j}} \end{bmatrix} = \begin{bmatrix} W_3 B_{F_1 j} & W_3 B_{F_2 j} \\ C_{F_1 j} & C_{F_2 j} \end{bmatrix}, \tag{41}$$
$$\begin{bmatrix} \tilde{A}_F & \tilde{B}_{F_j} \\ \tilde{M}_F & \tilde{D}_{F_j} \end{bmatrix} = \begin{bmatrix} W_3 A_F & W_3 B_{F j} \\ W_3 M_F & D_{F j} \end{bmatrix}, \tag{42}$$

with $j \in \{a, b\}$.
The following theorem gives the solutions to the design of the interval filter matrices such that the inequality $\underline{z}(t) \leq z(t) \leq \overline{z}(t)$, $\forall t \geq 0$ is satisfied and such that the effect of ω on the estimation error interval $\tilde{e}_z$ is minimized in the H_∞-norm criterion sense.

Theorem 2 *Consider the system* (8a)-(8d) *and assumptions 1-2, if there exist some real matrices* $P, W, Z, J, M, \tilde{\Sigma}_{11}, \tilde{\Sigma}_{12}, \tilde{\Sigma}_{21}, \tilde{\Sigma}_{22}, \tilde{\Psi}_{11}, \tilde{\Psi}_{12}, \tilde{\Psi}_{21}, \tilde{\Psi}_{22}, \tilde{\Xi}_{11}, \tilde{\Xi}_{12}, \tilde{\Xi}_{21}, \tilde{\Xi}_{22}, \tilde{\Pi}_{11}, \tilde{\Pi}_{12}, \tilde{A}_F, \tilde{M}_F, \tilde{B}_{F_a}, \tilde{B}_{F_b}, \tilde{B}_{F_{1a}}, \tilde{B}_{F_{1b}}, \tilde{B}_{F_{2a}}, \tilde{B}_{F_{2b}}, \tilde{C}_{F_{1a}}, \tilde{C}_{F_{1b}}, \tilde{C}_{F_{2a}}, \tilde{C}_{F_{2b}}, \tilde{D}_{F_a}, \tilde{D}_{F_b}$ *of adequate dimensions, some real scalars* $\kappa, \gamma > 0$ *and a M-matrix* W_3 *such that the following linear and bilinear constraints hold*

$$P = P^t > 0, \tag{43}$$
$$\underbrace{\begin{bmatrix} 0 & P & 0 & 0 \\ P & 0 & 0 & \tilde{C}^t \\ 0 & 0 & -I & \tilde{D}^t \\ 0 & \tilde{C} & \tilde{D} & -\gamma I \end{bmatrix}}_{\tilde{\Omega}} + \underbrace{\begin{bmatrix} -W & \tilde{A}_W & \tilde{B}_W & 0 \\ -Z & \tilde{A}_Z & \tilde{B}_Z & 0 \\ -J & \tilde{A}_J & \tilde{B}_J & 0 \\ -M & \tilde{A}_M & \tilde{B}_M & 0 \end{bmatrix}}_{\tilde{\Upsilon}} + \tilde{\Upsilon}^t < 0, \tag{44}$$
$$\tilde{B}_{F_{ia}}, \tilde{B}_{F_{ib}}, \tilde{C}_{F_{ia}}, \tilde{C}_{F_{ib}} \geq 0, \tag{45}$$
$$\tilde{B}_{F_a}, \tilde{B}_{F_b}, \tilde{D}_{F_a}, \tilde{D}_{F_b} \geq 0, \tag{46}$$
$$\tilde{C}_{F_{ia}} - \tilde{C}_{F_{ib}} \geq 0, \tag{47}$$
$$\tilde{A}_F + \kappa W_3 \geq 0, \tag{48}$$
$$(\tilde{D}_{F_a} - \tilde{D}_{F_b}) C T^\dagger = I_{n_z}, \tag{49}$$

where $\tilde{A}_W = W\overline{A}$, $\tilde{A}_Z = Z\overline{A}$, $\tilde{A}_J = J\overline{A}$, $\tilde{A}_M = M\overline{A}$, $\tilde{B}_W = W\overline{B}, \tilde{B}_Z = Z\overline{B}, \tilde{B}_J = J\overline{B}, \tilde{B}_M = M\overline{B}$, $\tilde{C}$ *and* $\tilde{D}$ *are defined in* (52)-(57), *with* $\mathbf{K} \in \{W, Z\}$ *and* $\mathbf{H} \in \{J, M\}$. *Here,* $\tilde{K}_{F_i} = \tilde{K}_{F_i a} - \tilde{K}_{F_i b}$ *with* $\tilde{K}_{F_i} \in \{\tilde{B}_{F_i}, \tilde{C}_{F_i}\}$ *and* $\tilde{X}_F = \tilde{X}_{F a} - \tilde{X}_{F_b}$ *with* $\tilde{X}_F \in \{\tilde{B}_F, \tilde{D}_F\}$ *and*

$$\begin{aligned} &\tilde{\Sigma}_{i1} = W_3\Sigma_{i1} = \tilde{R}_{i_a}Y^+ + \tilde{R}_{i_b}Y^-, && \tilde{\Xi}_{i1} = \tilde{C}_{F_{ib}}(-I_{n_f})^-, \\ &\tilde{\Sigma}_{i2} = W_3\Sigma_{i2} = \tilde{R}_{i_a}Y^- + \tilde{R}_{i_b}Y^+, && \tilde{\Xi}_{i2} = \tilde{C}_{F_{ia}}(-I_{n_f})^-, \\ &\tilde{\Psi}_{i1} = W_3\Psi_{i1} = \tilde{R}_{i_a}S^+ + \tilde{R}_{i_b}S^-, && \tilde{\Pi}_{11} = \tilde{D}_{F_a}(-F)^+ + \tilde{D}_{F_b}(-F)^-, \\ &\tilde{\Psi}_{i2} = W_3\Psi_{i2} = \tilde{R}_{i_a}S^- + \tilde{R}_{i_b}S^+, && \tilde{\Pi}_{12} = \tilde{D}_{F_a}(-F)^- + \tilde{D}_{F_b}(-F)^+, \end{aligned}$$

with $\tilde{R}_{i_a} = \begin{bmatrix} \tilde{B}_{F_{ia}} & \tilde{B}_{F a} \end{bmatrix}$, $\tilde{R}_{i_b} = \begin{bmatrix} \tilde{B}_{F_{ib}} & \tilde{B}_{F b} \end{bmatrix}$ *and* $i \in \{1, 2\}$. *Then, with an admissible realization of the state space of the filter, given by*

$$\begin{bmatrix} B_{F_1 j} & B_{F_2 j} \\ C_{F_1 j} & C_{F_2 j} \end{bmatrix} = \begin{bmatrix} W_3^{-1}\tilde{B}_{F_{1j}} & W_3^{-1}\tilde{B}_{F_{2j}} \\ \tilde{C}_{F_{1j}} & \tilde{C}_{F_{2j}} \end{bmatrix}, \tag{50}$$
$$\begin{bmatrix} A_F & B_{F j} \\ & D_{F j} \end{bmatrix} = \begin{bmatrix} W_3^{-1}\tilde{A}_F & W_3^{-1}\tilde{B}_{F_j} \\ & \tilde{D}_{F_j} \end{bmatrix}, \tag{51}$$

DOI: 10.13009/EUCASS2023-309

with $j \in \{a, b\}$, the state space (32) *is asymptotically stable with a performance criterion $||G_{\omega\to\tilde{e}_z}||_\infty \leq \gamma$.*

$$\tilde{A}_{\mathbf{K}} = \mathbf{K}\overline{A} = \begin{bmatrix} \mathbf{K}_1(A - BD_KC) & \mathbf{K}_1BC_K & 0 & 0 & 0 \\ -\mathbf{K}_2B_KC & \mathbf{K}_2A_K & 0 & 0 & 0 \\ (\tilde{B}_F - \tilde{M}_FD_K)C & \tilde{M}_FC_K & \tilde{A}_F & 0 & 0 \\ (\tilde{B}_{F_1} - \tilde{B}_F + \tilde{\Sigma}_{11}\tilde{D}_F - \tilde{\Sigma}_{12}\tilde{D}_F)C & 0 & \tilde{\Sigma}_{11}(\tilde{C}_{F_1} + \tilde{\Xi}_{11} - \tilde{\Xi}_{12}) - \tilde{\Sigma}_{12}(\tilde{C}_{F_2} + \tilde{\Xi}_{21} - \tilde{\Xi}_{22}) & \begin{matrix}\tilde{A}_F + \tilde{\Sigma}_{11}(\tilde{C}_{F_1} + \tilde{\Xi}_{11}) + \tilde{\Sigma}_{12}\tilde{\Xi}_{22} \\ -(\tilde{\Sigma}_{21}\tilde{\Xi}_{22} + \tilde{\Sigma}_{22}\tilde{C}_{F_1} + \tilde{\Sigma}_{22}\tilde{\Xi}_{11})\end{matrix} & \tilde{\Sigma}_{11}\tilde{\Xi}_{12} + \tilde{\Sigma}_{12}(\tilde{C}_{F_2} + \tilde{\Xi}_{21}) \\ (\tilde{B}_F - \tilde{B}_{F_2} + \tilde{\Sigma}_{21}\tilde{D}_F - \tilde{\Sigma}_{22}\tilde{D}_F)C & 0 & \tilde{\Sigma}_{21}(\tilde{C}_{F_2} + \tilde{\Xi}_{21} - \tilde{\Xi}_{22}) - \tilde{\Sigma}_{22}(\tilde{C}_{F_1} + \tilde{\Xi}_{11} - \tilde{\Xi}_{12}) & & \tilde{A}_F - \tilde{\Sigma}_{21}\tilde{C}_{F_2} - \tilde{\Sigma}_{21}\tilde{\Xi}_{21} - \tilde{\Sigma}_{22}\tilde{\Xi}_{12} \end{bmatrix}, \tag{52}$$

$$\tilde{B}_{\mathbf{K}} = \mathbf{K}\overline{B} = \begin{bmatrix} \mathbf{K}_1(E - BD_KF) & 0 & 0 \\ -\mathbf{K}_2B_KF & 0 & 0 \\ (\tilde{B}_F - \tilde{M}_FD_K)F & 0 & 0 \\ (\tilde{B}_{F_1} - \tilde{B}_F + \tilde{\Sigma}_{11}\tilde{D}_F - \tilde{\Sigma}_{12}\tilde{D}_F)F & (\tilde{\Sigma}_{11}\tilde{\Pi}_{11} + \tilde{\Sigma}_{12}\tilde{\Pi}_{12} + \tilde{\Psi}_{11}) & -(\tilde{\Sigma}_{11}\tilde{\Pi}_{12} + \tilde{\Sigma}_{12}\tilde{\Pi}_{11} + \tilde{\Psi}_{12}) \\ (\tilde{B}_F - \tilde{B}_{F_2} + \tilde{\Sigma}_{21}\tilde{D}_F - \tilde{\Sigma}_{22}\tilde{D}_F)F & -(\tilde{\Sigma}_{21}\tilde{\Pi}_{12} + \tilde{\Sigma}_{22}\tilde{\Pi}_{11} + \tilde{\Psi}_{22}) & (\tilde{\Sigma}_{21}\tilde{\Pi}_{11} + \tilde{\Sigma}_{22}\tilde{\Pi}_{12} + \tilde{\Psi}_{21}) \end{bmatrix}, \tag{53}$$

$$\tilde{A}_{\mathbf{H}} = \mathbf{H}\overline{A} = \begin{bmatrix} \mathbf{H}_1(A - BD_KC) - \mathbf{H}_2B_KC & \mathbf{H}_1BC_K + \mathbf{H}_2A_K & 0 & 0 & 0 \end{bmatrix}, \tag{54}$$

$$\tilde{B}_{\mathbf{H}} = \mathbf{H}\overline{B} = \begin{bmatrix} \mathbf{H}_1(E - BD_KF) - \mathbf{H}_2B_KF & 0 & 0 \end{bmatrix}, \tag{55}$$

$$\tilde{C} = \begin{bmatrix} 0 & (\tilde{C}_{F_1} - \tilde{\Xi}_{12} + \tilde{\Xi}_{11} - \tilde{C}_{F_2} - \tilde{\Xi}_{21} + \tilde{\Xi}_{22}) & (\tilde{C}_{F_1} + \tilde{\Xi}_{11} + \tilde{\Xi}_{22}) & (\tilde{\Xi}_{12} + \tilde{\Xi}_{21} + \tilde{C}_{F_2}) \end{bmatrix}, \tag{56}$$

$$\tilde{D} = \begin{bmatrix} 0 & (\tilde{\Pi}_{11} + \tilde{\Pi}_{12}) & -(\tilde{\Pi}_{12} + \tilde{\Pi}_{11}) \end{bmatrix}. \tag{57}$$

Proof 4 *Since the Theorem 2 has the linear variable changes defined in* (41)-(42)*, then it follows that the set of constraints (43)-(49) implies that*

$$P = P^t \succ 0, \tag{58}$$
$$\Omega + \Upsilon + \Upsilon^t \prec 0, \tag{59}$$
$$W_3B_{F_{ia}}, W_3B_{F_{ib}}, C_{F_{ia}}, C_{F_{ib}} \geq 0, \tag{60}$$
$$W_3B_{F_a}, W_3B_{F_b}, D_{F_a}, D_{F_b} \geq 0, \tag{61}$$
$$C_{F_{ia}} - C_{F_{ib}} \geq 0, \tag{62}$$
$$W_3A_F + \kappa W_3 \geq 0, \tag{63}$$
$$(D_{F_a} - D_{F_b})CT^\dagger = I_{n_z}. \tag{64}$$

Based on Lemma 1, the matrix A_F satisfies the Metzler condition (63) *since $W_3^{-1} \geq 0$. The matrices $B_{F_j}, B_{F_{ij}}, C_{F_{ij}}, D_{F_j}$. satisfy the decompositions* (24a)-(24b) *into two positive (non-unique) matrices thanks to the conditions* (60)-(61)*. Based on these decompositions, it becomes clear that* (62) *and* (64) *are the constraints mentioned in Section 5. By imposing the decomposition* (24a)-(24b) *of the filter matrices (i.e. $A_F, B_F, B_{F_i}, C_{F_i}$ and D_F), the LMI and BMI constraints* (59) *guarantee a H_∞ performance criterion for the closed-loop state space* (32) *such that $\dot{V}(\hat{x}) - \gamma||\omega||^2 + \frac{1}{\gamma}||\tilde{e}_z||^2 < 0$. Then, substituting the auxiliary variables W, Z, J, M by the partitioned structure defined in* (37)-(40) *and replacing the state space matrices (33)-(36) in* (2)*, we get finally* (44)*. Then the assertion $||G_{\omega\to\tilde{e}_z}||_\infty \leq \gamma$ is proved.*

7. Simulation results for the satellite FES

As a reminder, Microscope is a scientific satellite that was launched in 2016 with the primary goal of verifying the weak equivalence principle proposed by A. Einstein with unparalleled precision. To achieve this, a sophisticated control system, described by equations (7a)-(7b) and of dimension $n_k = 18$, has been meticulously designed. The objective of this controller is to maintain both the attitude and linear acceleration of the satellite at zero, ensuring optimal conditions

DOI: 10.13009/EUCASS2023-309

for testing the equivalence principle.[10] The control law incorporates compensation for disturbances, which are vital for accurate experimental results. The linear model, expressed by equations (6a)-(6c), and given by

$$A = \begin{bmatrix} 0 & 0 & 0.0039 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 \\ -0.0039 & 0 & 0 & 0 & 0 & 1.0000 \\ 0 & 0 & 0 & 0.0001 & -0.000 & -0.0020 \\ 0 & 0 & 0 & -0.0000 & 0.0000 & 0.0001 \\ 0 & 0 & 0 & 0.0015 & -0.0003 & 0.0001 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.0349 & -0.0007 & 0.0000 \\ -0.0007 & 0.0264 & -0.0009 \\ -0.0012 & 0.0000 & 0.0433 \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0349 & -0.0007 & 0.0000 & 0 & 0 & 0 \\ -0.0007 & 0.0264 & -0.0009 & 0 & 0 & 0 \\ -0.0012 & 0.0000 & 0.0433 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} I_3 & 0_3 \end{bmatrix}, \quad F = \begin{bmatrix} 0_3 & I_3 \end{bmatrix}, \quad T = \begin{bmatrix} I_3 & 0_3 \end{bmatrix},$$

is obtained by means of a first order approximation of the nonlinear equations presented in[10] around its equilibrium point $x^\star = \begin{pmatrix} 0 & 0 & 0 & 0 & -\omega_s & 0 \end{pmatrix}^t$. For simulation purpose, the initial condition for the dynamics (8a) is considered as $x(0) = x^\star$. An extensive analysis of the disturbance term $w(t)$, conducted through intensive simulations using the Functional Engineering Simulator, reveals the following bounds: $\underline{w} = -10^{-5}\begin{pmatrix} 3 & 5.5 & 5 & 5.583 & 7.289 & 3.620 \end{pmatrix}^t$ and $\overline{w} = 10^{-5}\begin{pmatrix} 3.5 & 5 & 4.5 & 0.899 & 2.943 & 3.142 \end{pmatrix}^t$.
As shown on Figure 1, the H_∞ interval filter is implemented in the Microscope mission navigation unit. It is followed by a data fusion algorithm to provide the estimate $\hat{\Theta} = \begin{bmatrix} \hat{\phi} & \hat{\theta} & \hat{\psi} \end{bmatrix}^t$ to the control unit. The fusion algorithm is given by

$$\hat{\Theta}(t) = \begin{bmatrix} R_\phi & 0 & 0 & 1 - R_\phi & 0 & 0 \\ 0 & R_\theta & 0 & 0 & 1 - R_\theta & 0 \\ 0 & 0 & R_\psi & 0 & 0 & 1 - R_\psi \end{bmatrix} \begin{bmatrix} \underline{\Theta}(t) \\ \overline{\Theta}(t) \end{bmatrix} \tag{65}$$

where R_ϕ, R_θ, R_ψ are determined so that $\hat{\Theta}(t)$ is an optimal estimate of $\Theta(t)$, in the l_1-norm sense.

In this section, the H_∞-filter design methodology previously described is applied to the satellite model to estimate the upper and lower bounds of the true attitude of satellite, i.e $\underline{z}(t) \leq \Theta(t) \leq \overline{z}(t)$. The interval filter is synthesized by solving the following optimization problem

$$\min \gamma \quad \text{s.t. } (43) - (49). \tag{66}$$

By using the Nonlinear SDP solver PENLAB[6] with the YALMIP toolbox for Matlab, optimization problem (66) can be applied to the LTI system closed-loop system augmented (8a)-(8d), with $\tilde{D}_{F_a} = I_3$ and $\tilde{D}_{F_b} = 0_3$ such that (49) holds. By solving the optimization problem (66) with the dimension $n_f \in \{1...8\}$, we obtain different results for the paramater κ and the H_∞ performance γ, with all the constraints (43)-(49) satisfied. These results are presented in Table 1.

order n_f	3	4	5	6	7	8
performance γ	1.4142152	1.4142150	1.4142156	1.4142157	1.4142156	1.4142155
parameter κ	75.3	64.4	51.7	97.1	99.6	43.0
LMI variables	1904	2056	2221	2399	2590	2794
$\overline{\phi} - \underline{\phi}\,[deg]$	$0.0742\ 10^{-3}$	$0.0710\ 10^{-3}$	$0.0741\ 10^{-3}$	$0.0741\ 10^{-3}$	$0.0741\ 10^{-3}$	$0.0742\ 10^{-3}$
$\overline{\theta} - \underline{\theta}\,[deg]$	$0.1174\ 10^{-3}$	$0.1150\ 10^{-3}$	$0.1173\ 10^{-3}$	$0.1173\ 10^{-3}$	$0.1173\ 10^{-3}$	$0.1174\ 10^{-3}$
$\overline{\psi} - \underline{\psi}\,[deg]$	$0.0706\ 10^{-3}$	$0.0700\ 10^{-3}$	$0.0705\ 10^{-3}$	$0.0705\ 10^{-3}$	$0.0705\ 10^{-3}$	$0.0705\ 10^{-3}$

Table 1: Results for different dimensions of n_f

No feasible solution has been found by the PENLAB solver for a filter of order $n_f = 1$ and $n_f = 2$. However, starting from the third order, the PENLAB solver successfully converges to an optimal solution for the optimization problem (66). Within the range of orders $n_f \in \{3...8\}$, the performance criterion γ shows slight variation, with differences on the order of a thousandth. The lowest H_∞ performance, indicated by the smallest value of γ, is obtained for the filter of

DOI: 10.13009/EUCASS2023-309

order $n_f = 4$. The convergence values of the envelope $\tilde{e}_z(t) = \overline{\Theta}(t) - \underline{\Theta}(t)$ associated with each order n_f are also presented in Table 1, where it can be observed that the smallest envelope is obtained for the filter of order $n_f = 4$. Considering the above analysis and taking into account the number of variables involved in the Linear Matrix Inequalities (LMIs), it is decided to implement the interval filter of order $n_f = 4$ within the navigation unit, to estimate the upper and lower bounds of the true attitude of satellite. From this order and from the admissible realization (50)-(51), we obtain the following matrices

$$W_3 = \begin{bmatrix} 7.53007 & -0.73606 & -0.73606 & -0.73606 \\ -0.73606 & 7.53007 & -0.73606 & -0.73606 \\ -0.73606 & -0.73606 & 7.53007 & -0.73606 \\ -0.73606 & -0.73606 & -0.73606 & 7.53007 \end{bmatrix}, \quad M_F = 10^{-5} \begin{bmatrix} 0.72382 & 0.56248 & 0.92487 \\ 0.72382 & 0.56248 & 0.92487 \\ 0.72382 & 0.56248 & 0.92487 \\ 0.72382 & 0.56248 & 0.92487 \end{bmatrix},$$

$$A_F = \begin{bmatrix} -55.8967 & 1.20821 & 1.20821 & 1.20821 \\ 1.20821 & -55.8967 & 1.20821 & 1.20821 \\ 1.20821 & 1.20821 & -55.8967 & 1.20821 \\ 1.20821 & 1.20821 & 1.20821 & -55.8967 \end{bmatrix},$$

$$B_F = 10^{-4} \left(\begin{bmatrix} 0.74682 & 0.74683 & 0.74683 \\ 0.74682 & 0.74683 & 0.74683 \\ 0.74682 & 0.74683 & 0.74683 \\ 0.74682 & 0.74683 & 0.74683 \end{bmatrix} - \begin{bmatrix} 0.74793 & 0.74793 & 0.74793 \\ 0.74793 & 0.74793 & 0.74793 \\ 0.74793 & 0.74793 & 0.74793 \\ 0.74793 & 0.74793 & 0.74793 \end{bmatrix} \right),$$

$$B_{F_1} = 10^{-4} \left(\begin{bmatrix} 0.69168 & 0.69168 & 0.69168 \\ 0.69168 & 0.69168 & 0.69168 \\ 0.69168 & 0.69168 & 0.69168 \\ 0.69168 & 0.69168 & 0.69168 \end{bmatrix} - \begin{bmatrix} 0.69169 & 0.69169 & 0.69169 \\ 0.69169 & 0.69169 & 0.69169 \\ 0.69169 & 0.69169 & 0.69169 \\ 0.69169 & 0.69169 & 0.69169 \end{bmatrix} \right),$$

$$B_{F_2} = 10^{-4} \left(\begin{bmatrix} 0.89524 & 0.89525 & 0.89525 \\ 0.89524 & 0.89525 & 0.89525 \\ 0.89524 & 0.89525 & 0.89525 \\ 0.89524 & 0.89525 & 0.89525 \end{bmatrix} - \begin{bmatrix} 0.89936 & 0.89935 & 0.89935 \\ 0.89936 & 0.89935 & 0.89935 \\ 0.89936 & 0.89935 & 0.89935 \\ 0.89936 & 0.89935 & 0.89935 \end{bmatrix} \right),$$

$$C_{F_1} = \begin{bmatrix} 0.005681 & 0.005681 & 0.005681 & 0.005681 \\ 0.005681 & 0.005681 & 0.005681 & 0.005681 \\ 0.005681 & 0.005681 & 0.005681 & 0.005681 \end{bmatrix} - \begin{bmatrix} 0.002840 & 0.002840 & 0.002840 & 0.002840 \\ 0.002840 & 0.002840 & 0.002840 & 0.002840 \\ 0.002840 & 0.002840 & 0.002840 & 0.002840 \end{bmatrix},$$

$$C_{F_2} = \begin{bmatrix} 0.005680 & 0.005680 & 0.005680 & 0.005680 \\ 0.005680 & 0.005680 & 0.005680 & 0.005680 \\ 0.005680 & 0.005680 & 0.005680 & 0.005680 \end{bmatrix} - \begin{bmatrix} 0.002841 & 0.002841 & 0.002841 & 0.002841 \\ 0.002841 & 0.002841 & 0.002841 & 0.002841 \\ 0.002841 & 0.002841 & 0.002841 & 0.002841 \end{bmatrix}.$$

Simulations are next performed to appreciate the performance of the designed interval filter. Figures 2 illustrates the results for a simulation corresponding to an orbital period of $8000s$. For a better assessment of the simulation results, zooms are also plotted. As expected, we note that the attitude angles $\phi(t), \theta(t), \psi(t)$ of the satellite belong to its associated interval $\overline{z}(t)$ and $\underline{z}(t)$, $\forall t \geq 0$. With the initial conditions of the filter given as $\overline{s}_f(0) = 10^{-2}\,(4\;4\;4\;4)^t$, $\underline{s}_f(0) = 10^{-2}\,(1\;1\;1\;1)^t$ and $x_f(0) = 0_{4\times1}$, $\overline{x}_f(0) = \overline{s}_f(0)$, $\underline{x}_f(0) = -\underline{s}_f(0)$, the distance $\tilde{e}_z$ converges to a value $c = 10^{-3}\begin{pmatrix}0.0710 & 0.1150 & 0.0700\end{pmatrix}^t$ with a reasonable transient behaviour.

Finally, thanks to the arbitrary dimension of the interval filter, the introduced slack matrices and degrees of freedom, and the H_∞ performance criterion, a significantly tighter interval estimation can be achieved compared to the approaches presented in the works[15],[20] as applied in our previous work[16] a tighter interval compared to the approaches[15],[20] as applied in our previous work,[16] can be observed on the attitude states. Furthermore, the fusion algorithm (65), which combines the upper and lower bounds $\overline{\Theta}, \underline{\Theta}$, delivered by the H_∞ interval filter, plays a crucial role in providing an optimal estimation of the attitude angles (as shown in Figure 2, indicated by the red line). The effectiveness of this algorithm, along with the proof of stability of the closed-loop system, encompassing the satellite dynamics, controllers, sensors, and interval filter, as rigorously established in Theorem 2, ensures accurate trajectory tracking. As a result, the true attitude angles of the satellite closely follow their desired references, approaching the ideal value of '0'. With the interval filter design proposed in this paper, the satellite is capable of maintaining its desired rotation around the Earth at a velocity of ω_0, while simultaneously spinning at a rate of ω_s around its y-axis. These capabilities are crucial for various space applications that require precise navigation and orientation control. Based on the above arguments and the evidence presented in Figure 2, we assert that the interval filter design represents a highly promising and practical solution for enhancing the performance of navigation units in space applications. The combination of tighter interval estimation, robust fusion algorithms, and proven stability ensures improved accuracy and reliability in satellite attitude

DOI: 10.13009/EUCASS2023-309

determination, contributing to the advancement of space exploration and satellite missions.

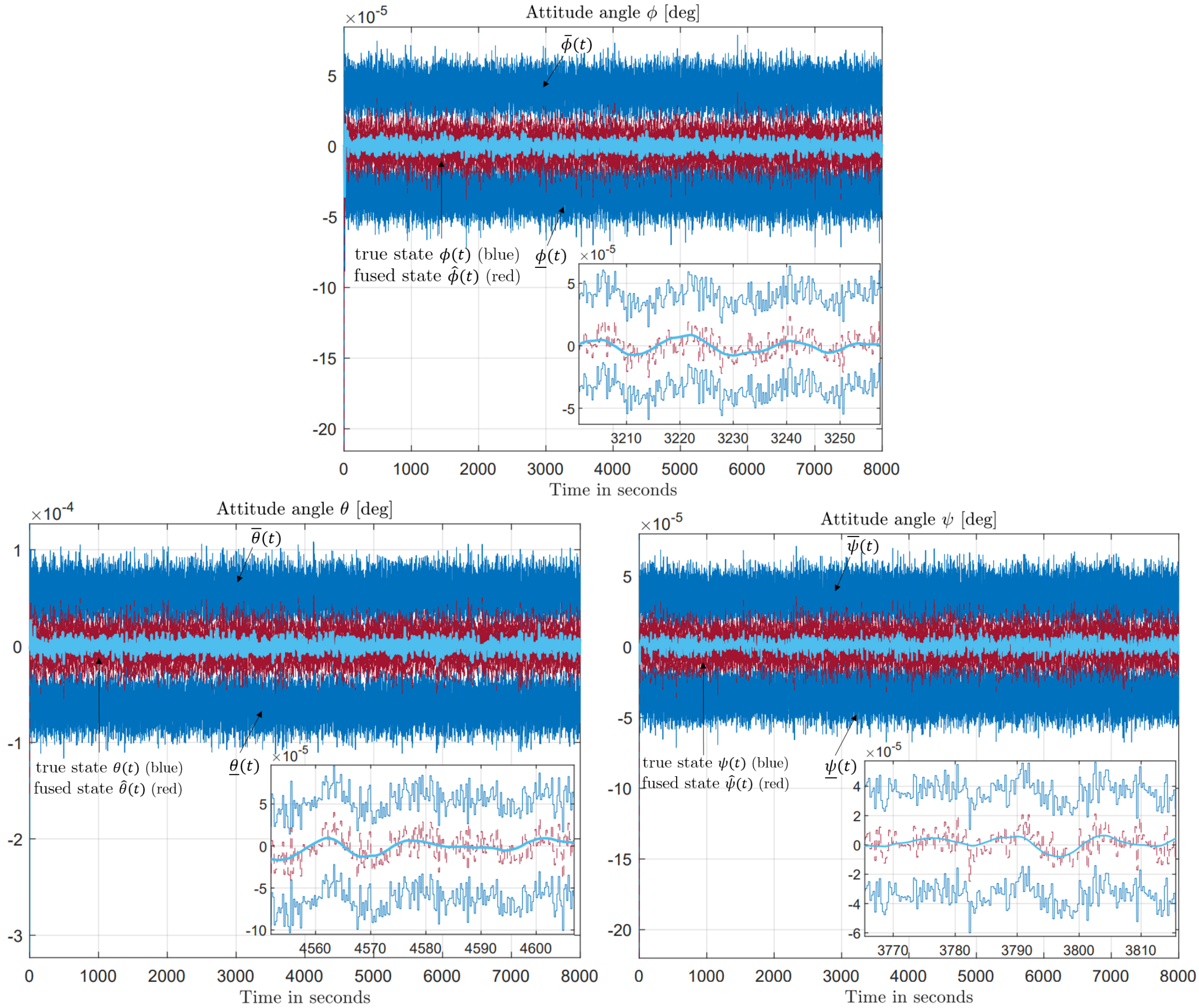

Figure 2: Interval estimation of the satellite attitude angles by applying the proposed method

8. Conclusion

This proposed paper deals with the development of a novel H_∞ interval filter and its application for accurate attitude estimation in satellite navigation units during space missions. The proposed approach offers several notable advantages compared to existing theories on interval observer design. One key advantage of the interval filter is its departure from the traditional interval observer structure. By introducing slack matrices and enabling the realization of a state space with arbitrary order, the filter allows for increased degrees of freedom in the estimation process. This increased flexibility enhances the interval filter's ability to handle uncertainties and disturbances more effectively. Another noteworthy feature is that this approach eliminates the need for explicit knowledge of a state transformation, which is typically required in other interval observer designs to satisfy the Meztler property for estimation error. By removing this requirement, the proposed interval filter achieves a significant reduction in conservatism, leading to more accurate and reliable estimates. The effectiveness of the proposed approach is validated through extensive simulations using a Functional Engineering Simulator (FES) of the satellite mission. The results demonstrate compelling performance metrics, including a notably reduced estimation interval for attitude angles and superior trajectory tracking capabilities for the attitude and acceleration control system.

DOI: 10.13009/EUCASS2023-309

References

[1] Vahid Aeinfar, Javad Askari, Arash Sadeghzadeh, and Mohsen Mojiri. Parameter dependent filter design for linear continuous-time uncertain systems with arbitrarily time-varying parameters in polytopic domains. *Signal Processing*, 165:83–89, December 2019.

[2] C. K. Chui and G. Chen. *Kalman filtering: with real-time applications*. Springer, Berlin, 4th ed edition, 2009.

[3] J.C. Doyle, K. Glover, P.P. Khargonekar, and B.A. Francis. State-space solutions to standard H_2 and H_∞ control problems. *IEEE Transactions on Automatic Control*, 34(8):831–847, 1989.

[4] N. Ellero, D. Gucik-Derigny, and D. Henry. Unknown input interval observer with H_∞ and D-stability performance. *IFAC-PapersOnLine*, 50(1):6251–6258, July 2017.

[5] Nicolas Ellero, David Gucik-Derigny, and David Henry. An unknown input interval observer for LPV systems under L_2 -gain and L_∞ -gain criteria. *Automatica*, 103:294–301, May 2019.

[6] Jan Fiala, Michal Kocvara, and Michael Stingl. Penlab: A matlab solver for nonlinear semidefinite optimization, 2013.

[7] Jose C. Geromel and Rafael C.D. Martins. A new method to H_2 robust filter design. *Linear Algebra and its Applications*, 430, 2009.

[8] Jose Claudio Geromel and Mauricio Carvalho de Oliveira. H_2 and H_∞ robust filtering for convex bounded uncertain systems. *IEEE Trans. Autom. Control.*, 46:100–107, 2001.

[9] J.L. Gouze, A. Rapaport, and M.Z. Hadj-Sadok. Interval observers for uncertain biological systems. *Ecological Modelling*, 133, 2000.

[10] D. Henry. Fault Diagnosis of Microscope Satellite Thrusters Using $H_\infty/H\_$ Filters. *Journal of Guidance, Control, and Dynamics*, 31(3):699–711, May 2008.

[11] Huijun Gao, J. Lam, Lihua Xie, and Changhong Wang. New approach to mixed H_2 / H_∞ filtering for polytopic discrete-time systems. *IEEE Transactions on Signal Processing*, 53:3183–3192, 2005.

[12] MÃ¡rcio J. Lacerda, Ricardo C.L.F. Oliveira, and Pedro L.D. Peres. Robust H_2 and H_∞ filter design for uncertain linear systems via lmis and polynomial matrices. *Signal Processing*, 91(5):1115–1122, 2011.

[13] Francoise Liorzou, Pierre Touboul, Manuel Rodrigues, Gilles Metris, Yves Andre, Joel Berge, Damien Boulanger, Stefanie Bremer, Ratana Chhun, Bruno Christophe, Pascale Danto, Bernard Foulon, Daniel Hagedorn, Emilie Hardy, Phuong-Anh Huynh, Claus Lammerzahl, Vincent Lebat, Meike List, Frank Loffler, Benny Rievers, Alain Robert, and Hanns Selig. MICROSCOPE instrument description and validation. *arXiv:2012.11232 [astro-ph, physics:gr-qc, physics:physics]*, 2020.

[14] Frederic Mazenc and Olivier Bernard. Interval observers for linear time-invariant systems with disturbances. *Automatica*, 47:140–147, 2011.

[15] Nacim Meslem, John Martinez, Nacim Ramdani, and Gildas Besancon. An interval observer for uncertain continuous time linear systems. *International Journal of Robust and Nonlinear Control*, 30:1886–1902, 2020.

[16] Irryhl Mohammedi, David Gucik-Derigny, and David Henry. Improvement of navigation unit in space applications using H_∞ interval observers: comparison of two approaches. In *2022 IEEE International Conference on Systems, Man, and Cybernetics (SMC)*, pages 1456–1461, 2022.

[17] George D. Poole and Thomas L. Boullion. A survey on m-matrices. *Siam Review*, 16:419–427, 1974.

[18] Pascal Prieur, Thomas Lienart, Manuel Rodrigues, Pierre Touboul, Troelz Denver, John Leif JÃ¸rgensen, Anastassia Malinnikova Bang, and Gillesz Metris. MICROSCOPE Mission: on-orbit assessment of the Drag-Free and Attitude Control System. 2017.

[19] Ye Wang, Vicenc Puig, and Gabriela Cembrano. Set-membership approach and Kalman observer based on zonotopes for discrete-time descriptor systems. *Automatica*, 93:435–443, July 2018.

DOI: 10.13009/EUCASS2023-309

[20] Zhenhua Wang, Cheng-Chew Lim, and Yi Shen. Interval observer design for uncertain discrete-time linear systems. *Systems & Control Letters*, 116, 2018.

[21] Feng Xu, Junbo Tan, Tarek RaÃ¯ssi, and Bin Liang. Design of optimal interval observers using set theoretic methods for robust state estimation. *International Journal of Robust and Nonlinear Control*, 30(9):3692–3705, June 2020.