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# $\mathcal{L}_1$ Adaptive Backstepping Control of Aircraft under Actuator Failures

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## Abstract

This paper presents a fault tolerant adaptive control scheme for an aircraft with actuator failures using $\mathcal{L}_1$ adaptive control. An $\mathcal{L}_1$ adaptive controller is proposed for a nonlinear, multi-input, affine-in-control system for the case when the closed-loop reference system is nonlinear. The proposed method is implemented in the control of an aircraft with actuator failures, where an ideal controller is designed a priori based on backstepping scheme to achieve asymptotic tracking of aerodynamic angles. Numerical simulation is conducted to demonstrate the performance of the proposed controller using the F/A-18 HARV model.

## 1. Introduction

Fault tolerant control has been an important research issue in flight control design for the past decades.[1] The objective of fault tolerant control is to design a controller that can accommodate the effect of various faults which can be fatal to the stability of the system. Fault tolerant control can be classified into two major categories, passive fault tolerant control and active fault tolerant control.[2] Passive fault tolerant control aims to design a controller that is robust to all deviations in the system within the bounds of the uncertainty.[3] Active fault tolerant control, on the other hand, aims to estimate the fault and reconfigure the controller online. Especially with the recent development of reference-based adaptive control schemes such as $\mathcal{L}_1$ adaptive control[4] and closed loop reference adaptive control,[5] adaptive control has become an appealing strategy in dealing with fault tolerant systems, because one can enforce the output of the plant under faults to follow that of the reference model by adjusting the control parameters through adaptation.

One of the challenging topics in fault tolerant control is the control of aircraft under actuator failures. Many related works decoupled the aircraft dynamics into longitudinal and lateral dynamics to simplify the problem.[6–8] However, asymmetric failures in actuators result in a complex coupling effect that cannot be handled by a simplified approach. Moreover, when an actuator failure occurs, the corresponding control input should be disconnected from the system. During this process, additive disturbance may be introduced, which may be time-varying and state dependent. For example, in the case of floating actuator failures, the control surface deflection follows an unknown angle such that the hinge moment becomes zero. This angle is mostly affected by the local angle of attack or side-slip angle, which is dependent on the relative velocity and angular rates. However, few research focus on the compensation of time-varying disturbances introduced by the actuator failures.

In this study, a nonlinear fault tolerant adaptive control scheme is proposed using the full aircraft dynamics to achieve good tracking of aerodynamic angles, i.e., aerodynamic roll angle, angle of attack, and sideslip angle even under actuator failure. An ideal controller would be one that achieves the same closed loop dynamics regardless of the actuator failure. To achieve fast adaptation without compromising robustness, $\mathcal{L}_1$ adaptive control is adopted. Although $\mathcal{L}_1$ adaptive control theory is limited to linear reference systems, similar results were also obtained for a class of nonlinear reference systems with single input using a first-order low-pass filter.[9] In this study, the pre-existing nonlinear $\mathcal{L}_1$ adaptive control theory is extended to treat systems with multiple inputs.

This paper is organized as follows. The problem formulation is given in Section 2. Section 3 presents the $\mathcal{L}_1$ adaptive controller for nonlinear affine-in-control systems with multiple inputs. Sections 4 and 5 deal with the fault tolerant

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adaptive control scheme for an aircraft with actuator failures. The design process of the virtual backstepping controller is introduced in Section 4, and simulation results conducted on the F/A-18 HARV model are discussed in Section 5. Section 6 presents the conclusion.

## 2. Problem Formulation

### 2.1 Preliminaries

**Notations** The euclidean norm of a vector and the induced norm of a matrix is denoted as $\|\cdot\|$. The truncated $\mathcal{L}_\infty$ norm of a vector signal $y(t) \in \mathbb{R}^n$ is defined as $\|y_\tau\|_{\mathcal{L}_\infty} = \sup_{0<t<\tau} \|y(t)\|$. The minimum and maximum singular values of a matrix is denoted by $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$.

**Projection Operator** Consider a convex compact set with a smooth boundary given by $\Omega_c = \{\theta \in \mathbb{R}^n | f(\theta) \leq c\}$, where $0 \leq c \leq 1$, and $f : \mathbb{R}^n \to \mathbb{R}$ is the smooth convex function defined as

$$f(\theta) = \frac{(\epsilon_\theta + 1)\,\|\theta\|^2 - \theta_{max}^2}{\epsilon_\theta \theta_{max}^2} \tag{1}$$

with positive constants $\theta_{max}$ and $\epsilon_\theta$. The projection operator is defined as

$$\text{Proj}(\theta, y) = \begin{cases} y - \frac{\nabla f}{\|\nabla f\|} \left\langle \frac{\nabla f}{\|\nabla f\|}, y \right\rangle f(\theta) & \text{if} \quad f(\theta) \geq 0 \cap \nabla f^\top y > 0 \\ y & \text{if} \quad \text{otherwise} \end{cases} \tag{2}$$

Note that $(\theta - \theta^*)^\top (\text{Proj}(\theta, y) - y) \leq 0$, for given $y \in \mathbb{R}^n$, $\theta^* \in \Omega_0$, and $\theta \in \Omega_1$. Also, for the system $\dot{\theta}(t) = \Gamma \text{Proj}(\theta(t), y(t))$ with any positive definite matrix $\Gamma$ and any piecewise continuous function $y(t) \in \mathbb{R}^n$, the projection operator ensures $\theta(t) \in \Omega_1$ for all $t \geq 0$, from any initial condition $\theta(0) \in \Omega_0$.[10]

**Lemma 1** *Consider a system*

$$\begin{aligned} \dot{z}(t) &= a(t)z(t) + b(t)v(t) \\ v(s) &= (I_{n\times n} - C(s))\sigma(s), z(0) = 0 \end{aligned} \tag{3}$$

*where $z(t) \in \mathbb{R}$ is the state and $v(t) \in \mathbb{R}^n$ is the input. The function $a(t) \in \mathbb{R}$ is continuous, $b(t) \in \mathbb{R}^{1\times n}$ is differentiable, $C(s) = \omega/(s + \omega) \times I_{n\times n}$ is a matrix low-pass filter, and $\sigma(t) :\in \mathbb{R}$ is a differentiable bounded input. Assume that $a(t)$, $b(t)$ and $\dot{b}(t)$ are bounded by $\|a(t)\| \leq p_1$, $\|b(t)\| \leq p_2$, and $\|\dot{b}(t)\| \leq p_3$ for all $t \in [0, \tau]$. Then,*

$$\|z_\tau\|_{\mathcal{L}_\infty} \leq \|\sigma_\tau\|_{\mathcal{L}_\infty} \int_0^\tau \left( p_2 e^{-\omega\varrho} + (p_1 p_2 + p_3) \int_0^\varrho e^{-\omega\lambda} \varphi(\varrho, \lambda) \right) d\lambda d\varrho \tag{4}$$

*where $\varphi(t, \tau) \geq 0$ is the state transition matrix of the system $\dot{z}(t) = a(t)z(t)$.*

*Furthermore, if $\|\dot{\sigma}_\tau\|_{\mathcal{L}_\infty}$ is also bounded, then*

$$\|z_\tau\|_{\mathcal{L}_\infty} \leq \|\sigma(0)\|\, p_2 \int_0^\tau e^{-\omega\varrho} \varphi(\tau, \varrho) d\varrho + \|\dot{\sigma}_\tau\|_{\mathcal{L}_\infty} p_2 \int_0^\tau \int_0^\varrho e^{-\omega\lambda} \varphi(\tau, \varrho) d\lambda d\varrho \tag{5}$$

### 2.2 Problem Statement

Consider the following nonlinear system:

$$\begin{aligned} \dot{x}(t) &= f(t, x(t)) + g(t, x(t))\tau, \quad x(0) = x_0 \\ \tau(t) &= \phi(t, x(t))u(t) + h(t, x(t)) \end{aligned} \tag{6}$$

where $x(t) \in \mathbb{R}^n$, $\tau(t) \in \mathbb{R}^l$, and $u(t) \in \mathbb{R}^m$ are the state, virtual control, and control input, $l \leq n$, $l \leq m$, $f : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times l}$, and $\phi : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^{l\times m}$ are known functions, and $h : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^l$ is an unknown function. Assume that $\phi(t, x(t))$ has full rank, there exists a virtual control law, $\tau(t) = k(t, x(t))$, and $f_m(t, x(t)) = f(t, x(t)) + g(t, x(t))k(t, x(t))$.

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### 2.3 Assumptions

**Assumption 1** *$f_m(t,x)$, $\frac{\partial f_m}{\partial x}(t,x)$, $g(t,x)$, $\phi(t,x)^\dagger$, and $h(t,x)$ are continuous, bounded, and lipschitz in $x$, uniformly in $t$, for all $t \in \mathbb{R}_0^+$ and all $x$ in a compact set.*

**Assumption 2** *$\frac{\partial g}{\partial t}(t,x)$, $\frac{\partial g}{\partial x}(t,x)$, $\frac{\partial h}{\partial t}(t,x)$, and $\frac{\partial h}{\partial x}(t,x)$ are bounded for all $t \in \mathbb{R}_0^+$ and all $x$ in a compact set.*

**Assumption 3** *There exists a function $\psi : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^{l \times n}$ such that $\psi(t,x)g(t,x) = I_{l\times l}$. Moreover, $\psi(t,x)$, $\frac{\partial \psi}{\partial t}(t,x)$, $\frac{\partial \psi}{\partial x}(t,x)$ are bounded for all $t \in \mathbb{R}_0^+$ and all $x$ in a compact set.*

**Assumption 4** *There exist positive constants $\gamma$, $c_1$, $c_2$, $c_3$, $c_4$, $c_5$, $c_6$ and a twice differentiable positive definite function $V : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}$ such that for all $t \geq 0$ and $e \in \{e \in \mathbb{R}^n | \, \|e\| \leq \gamma\}$:*

$$c_1 \|e\|^2 \leq V(t,e) \leq c_2 \|e\|^2 \tag{7}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial e} f_m(t,e) \leq -c_3 \|e\|^2 \tag{8}$$

$$\left\| \frac{\partial V}{\partial e} \right\| \leq c_4 \|e\|, \quad \left\| \frac{\partial^2 V}{\partial e^2} \right\| \leq c_5, \quad \left\| \frac{\partial^2 V}{\partial e \partial t} \right\| \leq c_6 \|e\| \tag{9}$$

**Assumption 5** *There exist positive constants $d_1$, $d_2$, $d_3$, $B_{\frac{\partial W}{\partial x}}$ and a differentiable positive definite function $W : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}$ such that for all $t \geq 0$ and $x \in \{x \in \mathbb{R}^n | \, \|x\| \leq \sqrt{d_2/d_1}\, \|x_0\| + \epsilon + \gamma\}$:*

$$d_1 \|x\|^2 \leq W(t,x) \leq d_2 \|x\|^2 \tag{10}$$

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial x} f_m(t,x) \leq -d_3 \|x\|^2 \tag{11}$$

$$\left\| \frac{\partial W}{\partial x} \right\| \leq B_{\frac{\partial W}{\partial x}} \tag{12}$$

Let us define $\rho$ and $\rho_{ref}$ as

$$\rho_{ref} = \sqrt{\frac{d_2}{d_1}} \|x_0\| + \epsilon \tag{13}$$

$$\rho = \rho_{ref} + \gamma \tag{14}$$

Then, from Assumptions 1, 2 and 3, we have for all $t \geq 0$ and all $x_1, x_2, x \in \{x \in \mathbb{R}^n | \, \|x\| \leq \rho\}$:

$$\|\Xi(t,x_1) - \Xi(t,x_2)\| \leq L_\Xi \|x_1 - x_2\| \tag{15}$$

$$\|\Psi(t,x)\| \leq B_\Psi \tag{16}$$

where $\Xi \in \{f_m, \frac{\partial f_m}{\partial x}, g, \phi^\dagger, h\}$, and $\Psi \in \{f_m, g, \phi^\dagger, h, \psi, \frac{\partial h}{\partial t}, \frac{\partial h}{\partial x}, \frac{\partial g}{\partial t}, \frac{\partial g}{\partial x}, \frac{\partial \psi}{\partial t}, \frac{\partial \psi}{\partial x}\}$.

**Assumption 6** *The constant $\gamma$ also satisfies*

$$2\gamma L_{\frac{\partial f_m}{\partial x}} < \frac{c_1 c_3}{c_2 c_4} \tag{17}$$

## 3. Nonlinear $\mathcal{L}_1$ Adaptive Controller

In this section, the structure of the $\mathcal{L}_1$ adaptive controller is introduced for the nonlinear multi-input multi-output system in Eq. (6). The structure of the proposed controller and its analysis discussed in the following subsections are similar to the single input version of the controller proposed in Wang and Hovakimyan.[9]

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### 3.1 Control Architecture

Consider the *state predictor*:

$$\dot{\hat{x}}(t) = f(t, x(t)) + g(t, x(t))\left(\phi(t, x(t))u + \hat{\sigma}(t)\right) + A_m\tilde{x}(t), \quad \hat{x}(0) = x_0 \tag{18}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the state prediction, $\tilde{x}(t) = \hat{x}(t) - x(t)$ is the prediction error, and $A_m \in \mathbb{R}^{n\times n}$ is a hurwitz matrix. The function $\hat{\sigma}(t) \in \mathbb{R}^l$ is the estimate of the unknown function $h(t, x(t))$ governed by the following projection-type *adaptive law*.

$$\dot{\hat{\sigma}}(t) = \Gamma\,\mathrm{Proj}(\hat{\sigma}(t), -g(t, x(t))^\top P\tilde{x}(t)), \quad \|\hat{\sigma}(0)\| \leq B_h \tag{19}$$

where $\Gamma \in \mathbb{R}^{l\times l}$ is the adaptation gain, and $P = P^\top > 0$ is the solution of the algebraic Lyapunov equation $A_m^\top P + PA_m + Q = 0$ for arbitrary $Q = Q^\top > 0$. Let us define $\theta_{max}$ and $\epsilon_\theta$ such that $B_\sigma = \theta_{max}$ and $B_h = \theta_{max}/\sqrt{\epsilon_\theta + 1}$, where $B_\sigma$ is a positive constant such that $B_\sigma > B_h$. The projection operator ensures that $\|\hat{\sigma}(t)\| \leq B_\sigma$ for all $t > 0$.

The *adaptive control law* is defined as

$$u(t) = \phi(t, x(t))^\dagger\left(k(t, x(t)) - \hat{\eta}(t)\right) \tag{20}$$

where $\hat{\eta}(s) = C(s)\hat{\sigma}(s)$, $C(s) = \omega/(s+\omega) \times I_{l\times l}$ is a low-pass filter matrix, and $\omega$ is the bandwidth of $C(s)$.

### 3.2 Performance Analysis

Consider the following non-adaptive version of the adaptive control system in Eqs. (6) and (20). In this study, this system is called the *reference system*:

$$\begin{aligned}
\dot{x}_{ref}(t) &= f_m(t, x_{ref}(t)) + g(t, x_{ref}(t))(-\eta_{ref}(t) + h(t, x_{ref}(t))), \quad x_{ref}(0) = x_0 \\
u_{ref}(t) &= \phi(t, x_{ref}(t))^\dagger\left(k(t, x_{ref}(t)) - \eta_{ref}(t)\right) \\
\eta_{ref}(s) &= C(s)\mathcal{L}\{h(t, x_{ref}(t))\}
\end{aligned} \tag{21}$$

In this subsection, under certain conditions, the followings will be shown: (a) the reference system is bounded, (b) the prediction error is bounded, and (c) the error between the real system and the reference system is bounded. First, let us show that the reference system is bounded.

*Condition 1* The constant $\omega$ satisfies the following condition.

$$\rho_{ref}^2 \geq \frac{W(0, x_0)}{d_1} + \delta_3(\omega) \tag{22}$$

where

$$\begin{aligned}
\delta_3(\omega) &= \frac{B_{\frac{\partial W}{\partial x}} B_g d_2}{d_1}\left(\frac{B_h}{\|d_3 - d_2\omega\|} + \frac{B_{h_{ref}}}{d_3\omega}\right) \\
B_{h_{ref}} &= B_{\frac{\partial h}{\partial t}} + B_{\frac{\partial h}{\partial x}}(B_{f_m} + \|I - C(s)\|_{\mathcal{L}_1} B_g B_h)
\end{aligned}$$

*Remark*: Since $\rho_{ref}^2 \geq \frac{W(0,x_0)}{d_1}$, and $\delta_3(\omega)$ approaches 0 as $\omega$ increases, Eq. (22) holds *if $\omega$ is large enough.*

**Lemma 2** *Suppose Assumptions 1, 2, 4, and 5 hold. If $\omega$ satisfies the inequality (22), then $\|x_{ref}\|_{\mathcal{L}_\infty} \leq \rho_{ref}$ and the system is uniformly ultimately bounded, i.e. there exist $T > 0$ and $\epsilon(\omega, T)$ such that $\|x_{ref}\| \leq \epsilon(\omega, T)$ for any $t \geq T$ where*

$$\epsilon(\omega, T) = \sqrt{\frac{e^{-\frac{d_3}{d_2}T} W(0, x_0)}{d_1} + \delta_3(\omega)} \tag{23}$$

Next, let us show the boundedness of the prediction error and the filtered estimate of the unknown function $h(t, x(t))$.

DOI: 10.13009/EUCASS2019-164



**Lemma 3** *Suppose Assumptions 1 and 2 hold. If there exists $\rho > 0$ such that $\|x(t)\| \le \rho$ for all $t \in [0, \tau]$, then*

$$\|\dot{x}(t)\| \le B_{\dot{x}} = B_{f_m} + (1 + \|C(s)\|_{\mathcal{L}_1})B_g B_\sigma \tag{24}$$

$$\|\dot{h}(t, x(t))\| \le B_{\dot{h}} = B_{\frac{\partial h}{\partial t}} + B_{\frac{\partial h}{\partial x}} B_{\dot{x}} \tag{25}$$

$$\|\dot{\psi}(t, x(t))\| \le B_{\dot{\psi}} = B_{\frac{\partial \psi}{\partial t}} + B_{\frac{\partial \psi}{\partial x}} B_{\dot{x}} \tag{26}$$

*for all $t \in [0, \tau]$.*

**Lemma 4** *Let $\tilde{x}(t) = \hat{x}(t) - x(t)$, $\tilde{\sigma}(t) = \hat{\sigma}(t) - h(t, x(t))$, and $\tilde{\eta}(s) = C(s)\tilde{\sigma}(s)$. Suppose Assumptions 1 and 2 hold. If there exists $\rho > 0$ such that $\|x(t)\| \le \rho$ for all $t \in [0, \tau]$, then*

$$\|\tilde{x}_\tau\|_{\mathcal{L}_\infty} \le \frac{\alpha}{\sqrt{\lambda_{min}(\Gamma)}} \tag{27}$$

$$\|\tilde{\eta}_\tau\|_{\mathcal{L}_\infty} \le \frac{\beta}{\sqrt{\lambda_{min}(\Gamma)}} \tag{28}$$

*where $\alpha = \sqrt{\frac{4B_\sigma{}^2}{\lambda_{min}(P)} + \frac{4B_\sigma B_{\dot{h}} \lambda_{max}(P)}{\lambda_{min}(Q)\lambda_{min}(P)}}$, and $\beta = \left(\|C(s)\|_{\mathcal{L}_1} (B_\psi \|A_m\| + B_{\dot{\psi}}) + \|C(s)s\|_{\mathcal{L}_1} B_\psi\right)\alpha$.*

The following lemma shows the lower bound on the adaptation gain for the performance bound on the error $x(t) - x_{ref}(t)$.

*Condition 2* There exists $T > 0$ such that

$$\mu \triangleq 2L_{\frac{\partial f_m}{\partial x}}\gamma + L_{\frac{\partial f_m}{\partial x}}\epsilon(\omega, T) < \frac{c_1 c_3}{c_2 c_4} \tag{29}$$

*Remark:* Note from Assumption 6 that, $L_{\frac{\partial f_m}{\partial x}}\gamma < \frac{c_1 c_3}{c_2 c_4}$, and therefore Eq. (29) holds *if $\omega$ and $T$ are large enough*.

*Condition 3* The constant $\omega$ satisfies the following condition.

$$\delta_1(\omega) + \delta_2(\omega) < c_1 \tag{30}$$

where

$$\delta_1(\omega) = L_h \frac{c_4 b_g \hat{\alpha} + \varrho(\rho_1 c_4 B_g + M)}{\hat{\alpha}\omega} \tag{31}$$

$$\delta_2(\omega) = \frac{B_h c_4 L_g \varrho}{\|\hat{\alpha} - \omega\|} + \frac{B_{\dot{h}_{ref}} c_4 L_g \varrho}{\hat{\alpha}\omega} \tag{32}$$

with $\rho_1 = \frac{c_3}{c_2} + \left(2L_{\frac{\partial f_m}{\partial x}}\rho + L_{\frac{\partial f_m}{\partial x}}\rho_{ref}\right)\frac{c_4}{c_1}$, $\hat{\alpha} = \frac{c_3}{c_2} - \mu\frac{c_4}{c_1} > 0$, $\varrho = \exp\left(c_4 L_{\frac{\partial f_m}{\partial x}}\rho_{ref} T / c_1\right)$, $M = c_5 B_g (B_g L_h + L_g B_h) \|I - C(s)\|_{\mathcal{L}_1} + c_6 B_g + c_5 B_g \left(L_{f_m} + L_{\frac{\partial f_m}{\partial x}}(2\gamma + \rho_{ref})\right) + c_4 B_{\dot{g}}$, $B_{\dot{g}} = B_{\frac{\partial g}{\partial t}} + B_{\frac{\partial g}{\partial x}} B_{\dot{x}}$, and $B_{\dot{x}} = B_{f_m} + (1 + \|C(s)\|_{\mathcal{L}_1})B_g B_h$.

**Lemma 5** *Let $x_{ref}(t) - x(t) = e(t)$. Suppose that Assumptions 1 to 6 hold. If there exists $\rho > 0$ such that $\|x(t)\| \le \rho$ for all $t \in [0, \tau]$, assuming inequalities (22), (29), and (30) hold, and the adaptation gain $\Gamma$ is selected large enough to satisfy*

$$\frac{\rho B_g(\omega c_4 + L_h B_g c_5)}{\omega\hat{\alpha}(c_1 - \delta_1(\omega) - \delta_2(\omega))} \frac{\beta}{\sqrt{\lambda_{min}(\Gamma)}} < \gamma_1 \tag{33}$$

*with $\gamma_1 < \gamma$, then we have $\|e_\tau\|_{\mathcal{L}_\infty} < \gamma_1$.*

Lemma 3 through 5 assumes the boundedness of $\|x(t)\|$. The following theorem shows that this can be relaxed.

**Theorem 1** *Suppose that Assumptions 1 to 6 hold. Assuming inequalities (22), (29), (30), and (33) hold, we have $\|x_{ref} - x\|_{\mathcal{L}_\infty} < \gamma_1$. Moreover, if $k(t, x(t))$ is bounded and locally Lipschitz in $x$, uniformly in $t$, then, $\|u_{ref} - u\|_{\mathcal{L}_\infty} \le \gamma_u$, where*

$$\gamma_u = B_{\phi^\dagger}(L_k + \|C(s)\|_{\mathcal{L}_1} L_h)\gamma_1 + L_{\phi^\dagger}\left(B_k + \|C(s)\|_{\mathcal{L}_1} B_h\right)\gamma_1 + B_{\phi^\dagger}\frac{\beta}{\sqrt{\lambda_{min}(\Gamma)}}$$

*and $B_k$ is the bound of $k(t, x(t))$, and $L_k$ is the lipschitz constant of $k(t, x(t))$ over $\{x \in \mathbb{R}^n \mid \|x\| \le \rho\}$.*

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### 3.3 Design Analysis

In this subsection, the closeness of the reference system and the design system will be shown. The design system is defined as

$$\begin{aligned} \dot{x}_{des}(t) &= f_m(t, x_{des}(t)), \quad x_{des}(0) = x_0 \\ u_{des}(t) &= \phi(t, x_{des}(t))^{\dagger} \left(k(t, x_{des}(t)) - h(t, x_{des}(t))\right) \end{aligned} \tag{34}$$

*Condition 4* There exists $\gamma_2 < \gamma$ such that

$$\frac{B_g \delta_2(\omega)}{L_g} < c_2 \gamma_2 \tag{35}$$

**Theorem 2** *Suppose that Assumptions 1 to 6 hold. Given a positive constant $\gamma_2 < \gamma$, if inequalities (22), (29), and (35) hold, we have $\|x_{ref} - x_{des}\|_{\mathcal{L}_\infty} < \gamma_2$. Moreover, if $k(t, x(t))$ is bounded and locally Lipschitz in x, uniformly in t, then,*

$$\|u_{ref}(t) - u_{des}(t)\| \leq B_{\phi^\dagger}(L_k + L_h)\gamma_2 + L_{\phi^\dagger} B_k \gamma_2 + B_{\phi^\dagger}\left(\|h(0, x_0)\|\, e^{-\omega t} + \frac{B_{\dot{h}_{ref}}(1 - e^{-\omega t})}{\omega}\right) \tag{36}$$

*where $B_k$ is the bound of $k(t, x(t))$, and $L_k$ is the lipschitz constant of $k(t, x(t))$ over $\{x \in \mathbb{R}^n \mid \|x\| \leq \rho\}$.*

**Proof** The proof is similar to the proof of Lemma 5, and is omitted due to the limited space.

## 4. Fault Tolerant Control of an Aircraft with Actuator Failures

In this section, the $\mathcal{L}_1$ adaptive controller proposed in the previous section is used to achieve aerodynamic angle tracking of an aircraft with actuator failures. The target aircraft model is the F/A-18 HARV. If the direct effect of control input in the aerodynamic angle dynamics is neglected, the aircraft system can be approximated as a strict-feedback form with a relative degree of two. First, a virtual controller that achieves asymptotic tracking of aerodynamic angles is introduced, and the system is reformulated to match the system defined in (6). Following the work of Seo and Kim,[11] a virtual control law $\tau(t) = k(t, x(t))$ is designed based on backstepping scheme. Finally, an $\mathcal{L}_1$ adaptive controller is formulated which estimates the time varying disturbances via adaptation, and cancels the disturbance within the bandwidth of the low-pass filter.

### 4.1 Aircraft Model

The aerodynamic angle dynamics, velocity roll angle $\mu$, sideslip angle $\beta$, and angle of attack $\alpha$, of the F/A-18 HARV model can be represented as

$$\dot{\xi}(t) = f_1(t, \xi(t)) + g_1(t, \xi(t))\zeta(t), \quad \xi(0) = \xi_0 \tag{37}$$

$$\dot{\zeta}(t) = f_2(t, \xi(t), \zeta(t)) + g_2(t, \xi(t), \zeta(t))\left(\Lambda u + (I - \Lambda)u_f(t, \xi(t), \zeta(t))\right), \quad \zeta(0) = \zeta_0 \tag{38}$$

where $\xi = [\mu\ \beta\ \alpha]^\top$ are the velocity roll angle, sideslip angle, and angle of attack, $\zeta = [p\ q\ r]^\top$ are body-axis roll, pitch, and yaw rates, $u = [\delta_{el}\ \delta_{er}\ \delta_a\ \delta_r]^\top$ are left elevator, right elevator, aileron, and rudder deflection, and $u_f(t, \xi(t), \zeta(t)) \in \mathbb{R}^4$ is an unknown function representing the input disturbance due to faults in the actuator. Also, let us define $\Gamma$ as follows,

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \tag{39}$$

which represents the fault status where $\lambda_i = 0$ means the loss of control and $\lambda_i = 1$ means no fault in the i'th actuator. Assume that $\Lambda$ is known, and $g_2(t, \xi, \zeta)\Lambda$ is always column-wise full rank. Note that the direct effect of control input in (37) is neglected in this study.

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### 4.2 Virtual Backstepping Controller

Following the work of Seo and Kim,[11] a virtual output tracking controller is designed via backstepping. Let $\xi_r(t) \in \mathbb{R}^3$ be a twice differentiable tracking reference, and let us define the tracking error $e_\xi = \xi - \xi_r$. Also, let us define a virtual control $\nu(t, e_\xi) \in \mathbb{R}^3$ such that $\zeta = \nu(t, e_\xi)$ in Eq. (37) renders $e_\xi$ uniformly asymptotically stable.

$$\nu(t, e_\xi(t)) = -g_1(t, \xi(t))^{-1}\left(f_1(t, \xi(t)) - \dot{\xi}_r(t) + Ke_\xi(t)\right) \tag{40}$$

where $K$ is some positive definite matrix. Let us also define $z(t) = \zeta(t) - \nu(t, e_\xi(t))$ and $x(t) = [e_\xi(t)^\top z(t)^\top]^\top$. Then, the system can be reformulated as follows,

$$\begin{aligned} \dot{x}(t) &= f(t, x(t)) + g(t, x(t))\tau(t) \\ \tau(t) &= \phi(t, x(t))u(t) + h(t, x(t)) \end{aligned} \tag{41}$$

where

$$f(t, x(t)) = \begin{pmatrix} Ke_\xi(t) + g_1(t, \xi(t))z(t) \\ f_2(t, \xi(t), \zeta(t)) - \left(\frac{\partial \nu}{\partial t} + \frac{\partial \nu}{\partial e_\xi}(Ke_\xi(t) + g_1(t, \xi(t))z(t))\right) \end{pmatrix}$$

with $g(t, x) = [0_{3\times3}\ I_{3\times3}]^\top$, $\phi(t, x) = g_2(t, \xi(t), \zeta(t))\Lambda$, and $h(t, x) = g_2(t, \xi(t), \zeta(t))(I - \Lambda)u_f(t, \xi(t), \zeta(t))$. It can be proven that the following control $\tau = k(t, x)$

$$k(t, x(t)) = -f_2(t, \xi(t), \zeta(t)) + \left(\frac{\partial \nu}{\partial t} + \frac{\partial \nu}{\partial e_\xi(t)}(Ke_\xi(t) + g_1(t, \xi(t))z(t))\right) - P_2^{-1}\left(g_1(t, \xi(t))^\top e_\xi(t) + Nz\right) \tag{42}$$

renders $x(t)$ asymptotically stable with

$$f_m(t, x(t)) = \begin{pmatrix} -Ke_\xi(t) + g_1(t, \xi(t))z(t) \\ -P^{-1}\left(g_1(t, \xi(t))^\top e_\xi(t) + Nz(t)\right) \end{pmatrix}$$

and a lyapunov function $U(t, x) = \frac{1}{2}e_\xi{}^\top e_\xi + \frac{1}{2}z^\top Pz$ satisfying Assumptions 4 and 5, given that $x_0 \in \Lambda_c^2$ where $\Lambda_c^2$ is defined in Seo et al.[11]
Once the virtual control law $\tau = k(t, x)$ is established, the $\mathcal{L}_1$ adaptive controller in (18), (19), and (20) can be adopted.

## 5. Numerical Simulation

In this section, the $\mathcal{L}_1$ adaptive fault tolerant controller proposed in this study is applied to the F/A-18 HARV model with actuator failures. Because the actual bounds of the functions in Assumption 1 and 2 are hard to calculate, the adaptation gain $\Gamma$ and the bandwidth $\omega$ were selected by trial and error. The controller parameters used in the simulations are summarized in Table 1.

Table 1: Controller parameters

| Parameter | Value |
|---|---|
| $K$ | $I_{3\times3}$ |
| $P$ | $I_{3\times3}$ |
| $N$ | $4\,I_{3\times3}$ |
| $A_m$ | $-5\,I_{3\times3}$ |
| $\theta_{max}$ | 10 |
| $\epsilon_\theta$ | 3 |
| $\Gamma$ | $1000\,I_{3\times3}$ |
| $\omega$ | 5 rad/s |

All actuator dynamics are assumed as a rate-limited first-order system with time constant $\tau_c = 0.1$s. It is assumed that the location of the faults can be obtained by a separate fault detection and isolation(FDI) strategy. The initial condition for all the simulations is the trim condition for a level flight at a airspeed of 130 m/s and an altitude of 3048m. The engine thrust is fixed to a constant trim value at the trim point.

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In all simulations, step commands were filtered through a rate-limited third-order low-pass filter of bandwidth 5 rad/s to generate the tracking reference. Step commands were given to $\beta(5^\circ)$ at $t = 10$s, $\alpha(10^\circ)$ at $t = 15$s, and $\mu(30^\circ)$ at $t = 20$s, consecutively. Two scenarios are considered in this study. In both scenarios, a floating actuator failure occurs at the right elevator at $t = 15$s. Since the F/A-18 HARV model has only four control surfaces, there can be only one actuator failure occurring at a given time in order for $\phi(t, x(t))$ to have full rank. In Scenario 1, the floating control surface is assumed to follow the negative of angle of attack. In Scenario 2, the failed control surface is assumed to follow an external sinusoidal signal of magnitude $5^\circ$ and frequency $\pi/2$ rad/s.

To demonstrate the performance of the proposed controller, a simulation was conducted using the controller without the adaptive portion, $u(t) = \phi(t, x(t))^\dagger k(t, x(t))$, in a nominal condition without any actuator fault. The results are shown in Fig. 1. It can be seen that the aerodynamic angles follow the reference signal without any performance degradation in a wide range of flight envelope. Fig. 2 shows the simulation results of scenario 1. The controller achieves good tracking of aerodynamic angles despite the actuator fault. In Fig. 2, $\eta$ represents the disturbance introduced by the actuator filtered through the low-pass filter, and $\hat{\eta}$ represents the filtered estimated disturbance from the adaptive law. It can be seen that the state predictor and the adaptive law provide almost perfect identification of the disturbance. Fig. 3 shows the simulation results of scenario 2. Similar to the result in scenario 1, the filtered disturbance is almost perfectly identified. However, in this scenario, there exists some degradation in performance, because the $\mathcal{L}_1$ adaptive controller only attempts to compensate for the disturbance within the bandwidth of the low-pass filter, and higher frequencies are not compensated. The spikes in disturbance around $t = 10$s, $t = 15$s and $t = 20$s shown in Figs. 2 and 3 are caused by the difference between the actuator command and the actual actuator response, which was not considered in the design of the controller. This means that the proposed $\mathcal{L}_1$ adaptive controller is also attempting to compensate for the unmodeled dynamics of the actuator. This suggests that that the unmodeled actuator dynamics should be taken into account, as done in the $\mathcal{L}_1$ adaptive control theory for linear reference systems, because increasing the bandwidth of the low-pass filter without considering the bandwidth of the actuator could result in an inadequate response of the system.

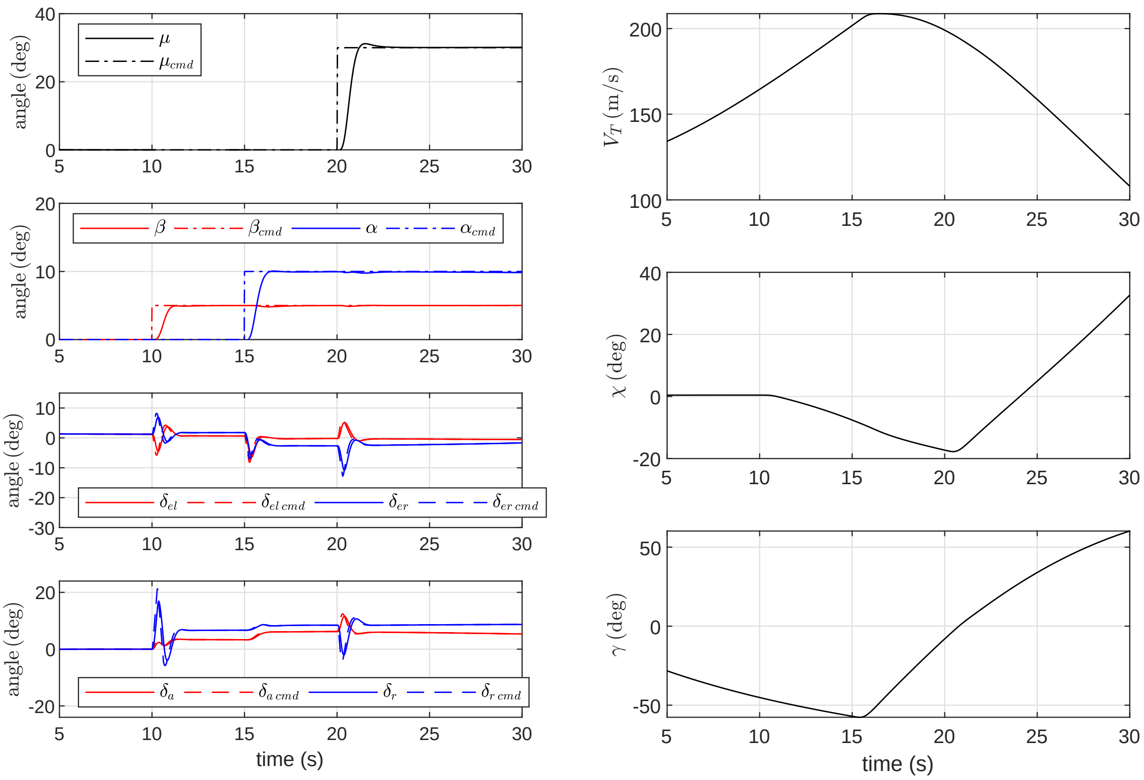


Figure 1: Nominal controller response

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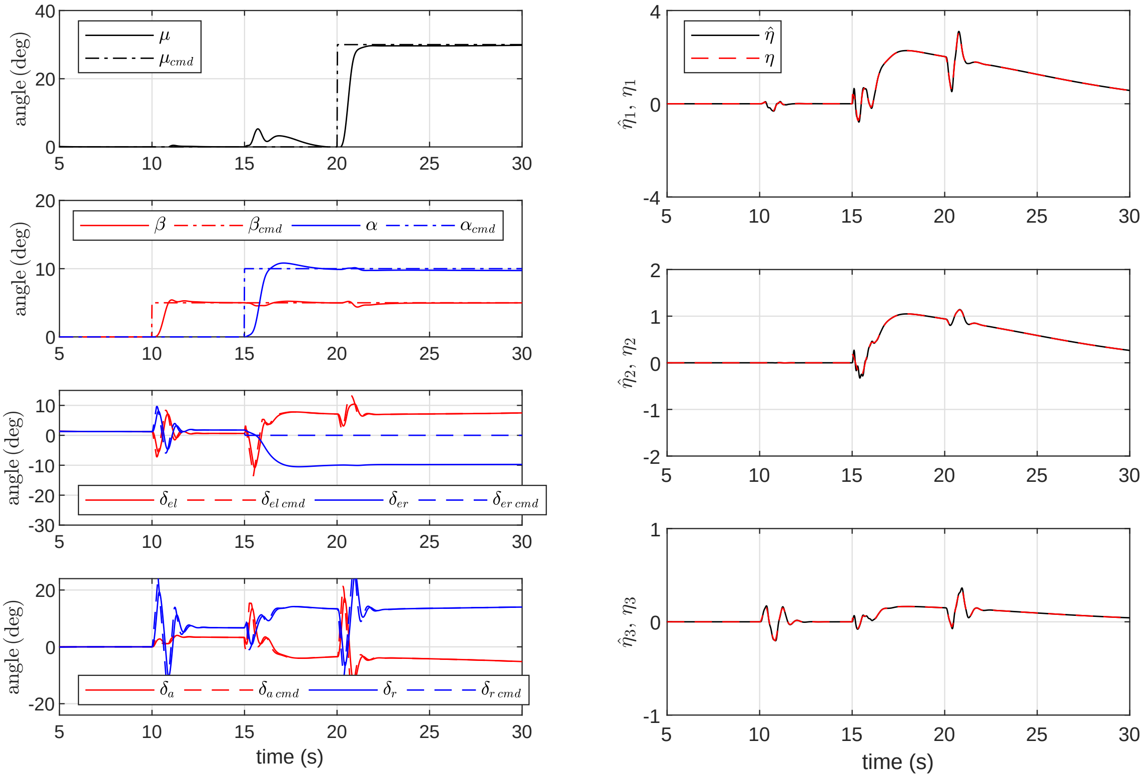


Figure 2: Floating control surface: negative angle of attack (Scenario 1)

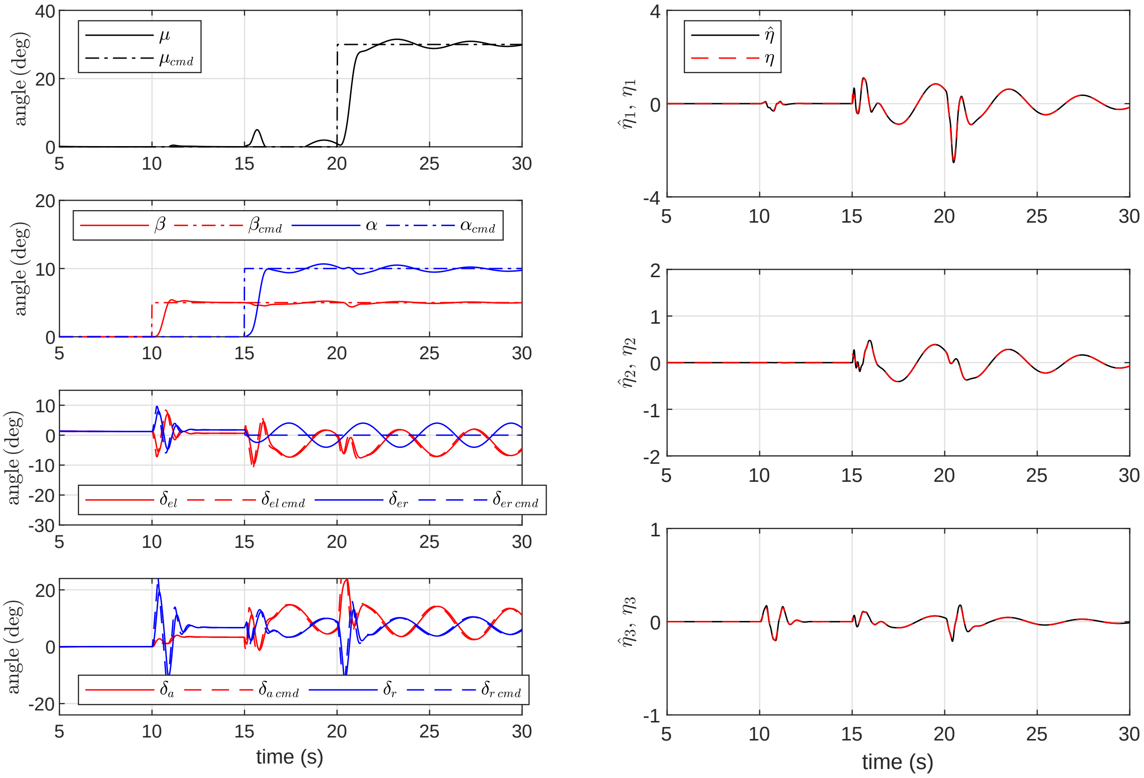


Figure 3: Floating control surface: sinusoldal response (Scenario 2)

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## 6. Conclusion

This paper proposed a fault tolerant adaptive control scheme for an aircraft with actuator failures using $\mathcal{L}_1$ adaptive control. A $\mathcal{L}_1$ adaptive controller for nonlinear reference systems was proposed for a family of nonlinear, multi-input, affine-in-control systems. Conditions for the performance bounds on the reference system, errors between the reference system and the real system, and errors between the reference system and the design system were derived. The proposed method was applied to achieve aerodynamic angle tracking of an aircraft with actuator failures. Finally, numerical simulations were conducted on the F/A-18 HARV model to demonstrate the performance of the proposed controller.

## Appendix

**Proof of Lemma 1** From the BIBO stability of LTV system, we have $\|z_\tau\|_{\mathcal{L}_\infty} = \|\mathcal{H}_\tau\|_{\mathcal{L}_1} \|\sigma_\tau\|_{\mathcal{L}_\infty}$, where $\mathcal{H}$ is the map from $\sigma$ to $z$.[12] Consider the impulse response matrix $q_z(t)$ of $z(t)$ and $q_v(t)$ of $v(t)$.

$$\begin{aligned}
q_v(t) &= I_{n\times n}(\delta(t) - \omega e^{-\omega t}) \\
q_z(t) &= \int_0^t \varphi(t,\lambda)b(\lambda)q_v(\lambda)d\lambda = \varphi(t,0)b(0) + \int_0^t (-\omega e^{-\omega\lambda})\varphi(t,\lambda)b(\lambda)d\lambda \\
&= \varphi(t,0)b(0) + e^{-\omega\lambda}\varphi(t,\lambda)b(\lambda)\Big|_{\lambda=0}^t - \int_0^t e^{-\omega\lambda}\left(-a(t)\varphi(t,\lambda)b(\lambda) + \varphi(t,\lambda)\dot{b}(\lambda)\right)d\lambda \\
&= e^{-\omega t}b(t) - \int_0^t e^{-\omega\lambda}\left(-a(t)\varphi(t,\lambda)b(\lambda) + \varphi(t,\lambda)\dot{b}(\lambda)\right)d\lambda \\
\|q_z(t)\| &\le e^{-\omega t}\|b(t)\| + \int_0^t e^{-\omega\lambda}\left(\|a(t)\|\,\varphi(t,\lambda)\,\|b(\lambda)\| + \varphi(t,\lambda)\left\|\dot{b}(\lambda)\right\|\right)dv \\
&\le p_2 e^{-\omega t} + (p_1p_2 + p_3)\int_0^t e^{-\omega\lambda}\varphi(t,\lambda)d\lambda
\end{aligned} \tag{A.1}$$

Therefore, we have (4) since $\|\mathcal{H}_\tau\|_{\mathcal{L}_1} = \int_0^\tau \|q_z(t)\|\,dt$. Furthermore, if $\|\dot{\sigma}_\tau\|_{\mathcal{L}_\infty}$ is also bounded, then the system can be rewritten as

$$\begin{aligned}
\dot{z}(t) &= a(t)z(t) + b(t)v(t), \quad z(0) = 0 \\
\dot{v}(t) &= -\omega v(t) + \dot{\sigma}(t), \quad v(0) = \sigma(0)
\end{aligned} \tag{A.2}$$

Solving this equation, we have

$$z(t) = \int_0^t \phi(t,\varrho)b(\varrho)\left(e^{-\omega\varrho}\sigma(0) + \int_0^\varrho e^{-\omega\lambda}\dot{\sigma}(\lambda)d\lambda\right)d\varrho \tag{A.3}$$

Therefore, we have (5). □

**Proof of Lemma 2** Suppose that the statement is not true. Since $\|x_{ref}(0)\| < \rho_{ref}$ and $x_{ref}(t)$ is continuous, there exists $\tau^* > 0$ such that $\|x_{ref}(\tau^*)\| = \rho_{ref}$ and $\|x_{ref}(t)\| < \rho_{ref}$ for all $t \in [0, \tau^*)$. By Assumption 5, over the time interval $[0, \tau^*]$, there exist a positive definite function $W(t, x_{ref})$ and constants $d_1, d_2, d_3$ such that

$$\begin{aligned}
\dot{W}(t, x_{ref}(t)) &= \frac{\partial W}{\partial t} + \frac{\partial W}{\partial x_{ref}}\dot{x}_{ref}(t) \\
&\le -d_3\|x_{ref}(t)\| + \frac{\partial W}{\partial x_{ref}}g(t, x_{ref}(t))(-\eta_{ref}(t) + h(t, x_{ref}(t))) \\
&\le -\frac{d_3}{d_2}W(t, x_{ref}(t)) + \frac{\partial W}{\partial x_{ref}}g(t, x_{ref}(t))(-\eta_{ref}(t) + h(t, x_{ref}(t)))
\end{aligned} \tag{A.4}$$

Solving this inequality, we have

$$W(t, x_{ref}(t)) \le e^{\frac{d_3}{d_2}t}W(0, x_0) + \int_0^t e^{\frac{d_3}{d_2}(t-\tau)}\frac{\partial W}{\partial x_{ref}}g(t, x_{ref}(t))(-\eta_{ref}(t) + h(t, x_{ref}(t)))d\tau \tag{A.5}$$

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The second term on the right-hand side of the preceding inequality is the solution to the following system.

$$\begin{aligned} \dot{z}(t) &= \frac{d_3}{d_2} z(t) + \frac{\partial W}{\partial x_{ref}} g(t, x_{ref}(t)) v(t) \\ v(s) &= (I_{l\times l} - C(s))\mathcal{L}(h(t, x_{ref}(t))), \quad z(0) = 0 \end{aligned} \tag{A.6}$$

where $\left\| \frac{\partial W}{\partial x_{ref}} g(t, x_{ref}(t)) \right\| \leq B_{\frac{\partial W}{\partial x}} B_g$ and $\|\dot{h}(t, x_{ref}(t))\| \leq B_{\dot{h}_{ref}}$ for all $t \in [0, \tau^*]$. Then, by Lemma 1, we have the following bound on $z(t)$:

$$\begin{aligned} \|z(t)\| &\leq B_{\frac{\partial W}{\partial x}} B_g B_h \int_0^t e^{-\omega\tau - \frac{d_3}{d_2}(t-\tau)} d\tau + B_{\frac{\partial W}{\partial x}} B_g B_{\dot{h}} \int_0^t \int_0^\tau e^{-\omega\lambda - \frac{d_3}{d_2}(t-\tau)} d\lambda d\tau \\ &= B_{\frac{\partial W}{\partial x}} B_g B_h \frac{e^{-\omega t} - e^{-\frac{d_3}{d_2} t}}{\frac{d_3}{d_2} - \omega} + \frac{B_{\frac{\partial W}{\partial x}} B_g B_{\dot{h}}}{\frac{d_3}{d_2} - \omega} \left( \frac{1 - e^{-\omega t}}{\omega} - \frac{1 - e^{-\frac{d_3}{d_2}}}{\frac{d_3}{d_2}} \right) \\ &\leq B_{\frac{\partial W}{\partial x}} B_g \left( \frac{B_h}{|d_3 - d_2\omega|} + \frac{B_{\dot{h}_{ref}}}{d_3\omega} \right) = d_1 \delta_3(\omega) \end{aligned} \tag{A.7}$$

for all $t \in [0, \tau^*]$, which implies that

$$\|x_{ref}(t)\| \leq \frac{e^{-\frac{d_3}{d_2} t} W(0, x_0)}{d_1} + \delta_3(\omega) \tag{A.8}$$

holds for all $t \in [0, \tau^*]$. Therefore, $\rho_{ref}^2 = \|x_{ref}(\tau^*)\| < \frac{W(0,x_0)}{d_1} + \delta_3(\omega)$, which contradicts (22). Thus, $\|x_{ref}\|_{\mathcal{L}_\infty} \leq \rho_{ref}$, and from (A.8), the system is also uniformly ultimately bounded. □

**Proof of Lemma 3** The system dynamics (6) with controller (20) leads to the following equation.

$$\dot{x}(t) = f_m(t, x(t)) + g(t, x(t))(h(t, x(t)) - \hat{\eta}(t)) \tag{A.9}$$

where $\|\hat{\eta}(t)\| \leq \|C(s)\| \, \|\hat{\sigma}\|$, and $\|\hat{\sigma}\| \leq B_\sigma$ is guaranteed by the projection operator. Thus, (24) is satisfied. Also,

$$\begin{aligned} \dot{h}(t, x(t)) &= \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} \dot{x}(t) \\ \dot{\psi}(t, x(t)) &= \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x} \dot{x}(t) \end{aligned} \tag{A.10}$$

which leads to the inequailties (25) and (26). □

**Proof of Lemma 4** The error dynamics between the real system and the state predictor is given as follows,

$$\dot{\tilde{x}}(t) = A_m \tilde{x}(t) + g(t, x(t))\tilde{\sigma}(t), \quad \tilde{x}(0) = 0 \tag{A.11}$$

Consider the following lyapunov candidate function.

$$U(t) = \tilde{x}(t)^\top P \tilde{x}(t) + \tilde{\sigma}(t)^\top \Gamma^{-1} \tilde{\sigma}(t) \tag{A.12}$$

The time derivative of $U(t)$ can be written as

$$\dot{U}(t) = -\tilde{x}(t)^\top Q \tilde{x}(t) + 2\tilde{x}(t)^\top P g(t, x(t))\tilde{\sigma}(t) + 2\tilde{\sigma}(t)^\top \Gamma^{-1} \dot{\hat{\sigma}}(t) + 2\tilde{\sigma}(t)^\top \Gamma^{-1} \dot{h}(t, x(t)) \tag{A.13}$$

Substituting (20) leads to

$$\dot{U}(t) \leq -\tilde{x}(t)^\top Q \tilde{x}(t) + 2\tilde{\sigma}(t)^\top \Gamma^{-1} \dot{h}(t, x(t)) \tag{A.14}$$

Using the bound on $\dot{h}(t, x(t))$ from Lemma 3, and $\|\tilde{\sigma}(t)\| = \|\hat{\sigma}(t) - h(t, x(t))\| \leq 2B_\sigma$, we have

$$\dot{U}(t) \leq -\lambda_{min}(Q) \, \|\tilde{x}(t)\| + \frac{4B_\sigma B_{\dot{h}}}{\lambda_{min}(\Gamma)} \tag{A.15}$$

which implies that $\|\tilde{x}(t)\| \leq \frac{4B_\sigma B_{\dot{h}}}{\lambda_{min}(Q)\lambda_{min}(\Gamma)}$ or $\dot{U}(t) \leq 0$. Thus, we have (27).
Now, consider $\|\tilde{\eta}_\tau\|_{\mathcal{L}_\infty}$. From (A.11) and Assumption 3, we have $\psi(t, x(t))\dot{\tilde{x}}(t) = \psi(t, x(t))A_m\tilde{x}(t) + \tilde{\sigma}(t)$, which implies

$$\tilde{\sigma}(t) = \frac{d}{dt} \left( \psi(t, x(t))\tilde{x}(t) \right) - \dot{\psi}(t, x(t))\tilde{x}(t) - \psi(t, x(t))A_m\tilde{x}(t) \tag{A.16}$$

Therefore,

$$\|\tilde{\eta}_\tau\|_{\mathcal{L}_\infty} \leq \|C(s)s\|_{\mathcal{L}_1} B_\psi \|\tilde{x}_\tau\|_{\mathcal{L}_\infty} + \|C(s)\|_{\mathcal{L}_1} B_{\dot{\psi}} \|\tilde{x}_\tau\|_{\mathcal{L}_\infty} + \|C(s)\|_{\mathcal{L}_1} B_\psi \|A_m\| \|\tilde{x}_\tau\|_{\mathcal{L}_\infty} \tag{A.17}$$

which leads to (28). □

**Proof of Lemma 5** The error dynamics between the real system and the reference system is given as follows,

$$\dot{e}(t) = f_m(t, e(t)) + \Delta(t, e(t)) + \Phi_1(t, x(t)) + \Phi_2(t, x(t)) \tag{A.18}$$

where

$$\begin{aligned} \Delta(t, e(t)) &= f_m(t, x_{ref}(t) + e(t)) - f_m(t, x_{ref}(t)) - f_m(t, e(t)) \\ \Phi_1(t, x(t)) &= g(t, x(t))(\eta(t) - \hat{\eta}(t)) \\ \Phi_2(t, x(t)) &= g(t, x(t))\left(h(t, x(t)) - \eta(t) - h(t, x_{ref}(t)) + \eta_{ref}(t)\right) \\ &\quad + \left(g(t, x(t)) - g(t, x_{ref}(t))\right)\left(h(t, x_{ref}(t)) - \eta_{ref}(t)\right) \end{aligned} \tag{A.19}$$

Suppose that the statement is not true. Since $\|e(0)\| = 0$ and $e(t)$ is continuous, there exists $\tau^* > 0$ such that $\|e(\tau^*)\| = \gamma_1$ and $\|e(t)\| = \gamma_1$ for all $t \in [0, \tau^*)$. Consider the system during the time interval $[0, \tau^*]$. By the mean value theorem, we have

$$\Delta(t, e(t)) = \frac{\partial f_m}{\partial x}\left(t, x_{ref}(t) + \lambda_1(t)e(t)\right)e(t) - \frac{\partial f_m}{\partial x}\left(t, \lambda_2(t)e(t)\right)e(t) \tag{A.20}$$

Therefore, the following bounds hold:

$$\begin{aligned} \|\Delta(t, e(t))\| &\leq 2L_{\frac{\partial f_m}{\partial x}} \|e(t)\|^2 + L_{\frac{\partial f_m}{\partial x}} \|x_{ref}(t)\| \|e(t)\| \\ \|\Phi_1(t, x(t))\| &\leq \frac{B_g \beta}{\sqrt{\lambda_{min}(\Gamma)}} \end{aligned} \tag{A.21}$$

By Assumption 4, over the time interval $[0, \tau^*]$, there exist a positive definite function $V(t, e(t))$ and constants $c_1$, $c_2$, $c_3$, $c_4$ such that

$$\begin{aligned} \dot{V}(t) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial e}\dot{e}(t) \\ &\leq -c_3 \|e(t)\|^2 + \left\|\frac{\partial V}{\partial e}\right\| \|\Delta(t, e(t))\| + \left\|\frac{\partial V}{\partial e}\right\| \|\Phi_1(t, x(t))\| + \frac{\partial V}{\partial e}\Phi_2(t, x(t)) \\ &\leq -c_3 \|e(t)\|^2 + c_4\kappa(t) \|e(t)\|^2 + \frac{c_4 B_g \beta \gamma_1}{\sqrt{\lambda_{min}(\Gamma)}} + \frac{\partial V}{\partial e}\Phi_2(t, x(t)) \end{aligned} \tag{A.22}$$

where $\kappa(t) = 2L_{\frac{\partial f_m}{\partial x}}\gamma_1 + L_{\frac{\partial f_m}{\partial x}} \left\|x_{ref}(t)\right\|$. Solving the differential inequality with $V(0, e(0)) = 0$ results

$$V(t, e(t)) \leq \int_0^t \varphi(t, \tau)\left(\frac{c_4 B_g \beta \gamma_1}{\sqrt{\lambda_{min}(\Gamma)}} + \frac{\partial V}{\partial e}\Phi_2(\tau, x(\tau))\right)d\tau \tag{A.23}$$

where $\varphi(t, \tau) = exp\left(-\frac{c_4}{c_2}(t - \tau) + \frac{c_4}{c_1}\int_\tau^t \kappa(\lambda)d\lambda\right)$.
Let us consider $\kappa(t)$. From Lemma 2, there exist $\epsilon(\omega, T)$ and $T > 0$ such that $\left\|x_{ref}(t)\right\| \leq \rho_{ref}$ for all $t > 0$ and $\left\|x_{ref}(t)\right\| \leq \epsilon(\omega, T)$ for all $t > T$. This implies

$$\int_0^t \|x_{ref}(\tau)\| d\tau \leq \epsilon(\omega, T)t + \rho_{ref}T \tag{A.24}$$

and

$$\int_0^t \kappa(\tau)d\tau \leq \mu t + L_{\frac{\partial f_m}{\partial x}}\rho_{ref}T \tag{A.25}$$

where $\mu$ is defined in Condition 2. Therefore, we have the following bound on $\varphi(t, \tau)$:

$$\varphi(t, \tau) \leq e^{-\hat{\alpha}(t-\tau)}\varrho \tag{A.26}$$

where $\hat{\alpha}$ and $\varrho$ are defined in Condition 3. Returning to the inequality (A.23), we have

$$V(t, e(t)) \leq \frac{\varrho}{\hat{\alpha}} \frac{c_4 B_g \beta \gamma_1}{\sqrt{\lambda_{min}(\Gamma)}} + \int_0^t \varphi(t, \tau)\frac{\partial V}{\partial e}\Phi_2(\tau, x(\tau))d\tau \tag{A.27}$$

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Note that, with the definition of $\Phi_2(t, x(t))$, the second term in the right-hand side of Eq.(A.27) is the sum of the states of two linear systems, $z_1(t) + z_2(t)$ given by

$$\begin{aligned} \dot{z}_1(t) &= \left(\frac{c_4\kappa_4(t)}{c_1} - \frac{c_3}{c_2}\right) z_1(t) + \frac{\partial V}{\partial e} g(t, x(t))\zeta_1(t) \\ \zeta_1(s) &= (I_{l\times l} - C(s))\mathcal{L}\{h(t, x(t)) - h(t, x_{ref}(t))\} \end{aligned} \tag{A.28}$$

and

$$\begin{aligned} \dot{z}_2(t) &= \left(\frac{c_4\kappa_4(t)}{c_1} - \frac{c_3}{c_2}\right) z_2(t) + \frac{\partial V}{\partial e}\left(g(t, x(t)) - g(t, x_{ref}(t))\right)\zeta_2(t) \\ \zeta_1(s) &= (I_{l\times l} - C(s))\mathcal{L}\{h(t, x_{ref}(t))\} \end{aligned} \tag{A.29}$$

The following bounds can be verified for all $t \in [0, \tau^*]$.

$$\begin{aligned} &\left\|\frac{c_4\kappa_4(t)}{c_1} - \frac{c_3}{c_2}\right\| \le \rho_1 && \left\|\frac{\partial V}{\partial e} g(t, x(t))\right\| \le c_4 B_g \gamma_1 \\ &\left\|\frac{d}{dt}\left(\frac{\partial V}{\partial e} g(t, x(t))\right)\right\| \le M\gamma_1 + \frac{c_5 B_g^2 \beta}{\lambda_{min}(\Gamma)} && \left\|h(t, x(t)) - h(t, x_{ref}(t)\right\| \le L_h \gamma_1 \\ &\left\|\frac{\partial V}{\partial e}\left(g(t, x(t)) - g(t, x_{ref}(t))\right)\right\| \le c_4 L_g \gamma_1^2 && \left\|\dot{h}(t, x_{ref}(t))\right\| \le B_{\dot{h}_{ref}} \end{aligned} \tag{A.30}$$

where $\rho_1$ and $M$ are defined in Condition 3, and $B_{\dot{h}_{ref}}$ is defined in Condition 1. Applying Lemma 1 to (A.28) and (A.29), similarly to Lemma 2, the following bounds can be obtained, for all $t \in [0, \tau^*)$.

$$\begin{aligned} \|\dot{z}_1(t)\| &\le \gamma_1^2 \delta_1(\omega) + \gamma_1 \frac{\theta_1}{\lambda_{min}(\Gamma)} \\ \|\dot{z}_2(t)\| &\le \gamma_1^2 \delta_2(\omega) \end{aligned} \tag{A.31}$$

where $\theta_1 = \frac{L_h c_5 B_g^2 \beta \varrho}{\hat{\alpha}\omega}$, and $\delta_1(\omega)$ and $\delta_2(\omega)$ are defined in Condition 3. Let $\theta_2 = \frac{L_h c_4 B_g \beta}{\hat{\alpha}}$. Then, returning once again to the inequality (A.27), we have

$$V(t, e(t)) \le \frac{\gamma_1(\theta_1 + \theta_2)}{\sqrt{\lambda_{min}(\Gamma)}} + (\delta_1(\omega) + \delta_2(\omega))\gamma_1^2 \tag{A.32}$$

for all $t \in [0, \tau^*]$. This implies that

$$c_1\gamma_1^2 \le \frac{\gamma_1(\theta_1 + \theta_2)}{\sqrt{\lambda_{min}(\Gamma)}} + (\delta_1(\omega) + \delta_2(\omega))\gamma_1^2 \tag{A.33}$$

which contradicts (35). Therefore, $\|e_\tau\|_{\mathcal{L}_\infty} < \gamma_1$. □

**Proof of Theorem 1** Suppose that the statement is not true. Since $\|x(0)\| < \rho$ and $x(t)$ is continuous, there exists $\tau^* > 0$ such that $\|x(\tau^*)\| = \rho$ and $\|x(t)\| < \rho$ for all $t \in [0, \tau^*)$. Consider the time interval $[0, \tau^*]$. Let $x_{ref}(t) - x(t) = e(t)$. According to Lemma 5, $\|e(t)\| < \gamma_1$ and since $\left\|x_{ref}(t)\right\| \le \rho_{ref}$, we have $\|x(t)\| < \rho_{ref} + \gamma_1$ for all $t \in [0, \tau^*]$, which contradicts the assumption $\|x(\tau^*)\| = \rho$. Hence, according to Lemma 5, we have $\|e_\tau\|_{\mathcal{L}_\infty} < \gamma_1$.

Also, from the definition of $u_{ref}(t)$ and $u(t)$ in (21) and (20), we have

$$\begin{aligned} u_{ref}(t) - u(t) &= \phi(t, x_{ref}(t))^\dagger\left(k(t, x_{ref}(t)) - \eta_{ref}(t)\right) - \phi(t, x(t))^\dagger\left(k(t, x(t)) - \hat{\eta}(t)\right) \\ &= \phi(t, x_{ref}(t))^\dagger\left(k(t, x_{ref}(t)) - k(t, x(t))\right) \\ &- \phi(t, x_{ref}(t))^\dagger\left(\eta_{ref}(t) - \eta(t) + \eta(t) - \hat{\eta}(t)\right) \\ &+ \left(\phi(t, x_{ref}(t))^\dagger - \phi(t, x(t))^\dagger\right)\left(k(t, x_{ref}(t)) - \eta_{ref}(t)\right) \end{aligned} \tag{A.34}$$

Therefore,

$$\left\|u_{ref}(t) - u(t)\right\| \le B_{\phi^\dagger} L_k \gamma_1 + B_{\phi^\dagger}\left(\|C(s)\| L_h \gamma_1 + \frac{\beta}{\sqrt{\lambda_{min}(\Gamma)}}\right) + L_{\phi^\dagger}\gamma_1\left(B_k + \|C(s)\| B_h\right) \tag{A.35}$$

for all $t > 0$. □

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