THEORY OF NAVIGATION ERRORS FOR A STRAP DOWN INERTIAL NAVIGATOR

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Navigation errors occur in present day strap down navigation systems as well as they intervened previously in systems using gimbaled platforms.

These errors come on the one hand from erroneous initialization of the navigation solution. They arise on the other hand from IMU (Inertial Measurement Unit) measurement errors and from earth gravity modeling.

A strap down IMU is made of a set of accelerometers and gyrometers which measure specific acceleration and body angular rate along the three directions of a body fixed reference frame. IMU measurement errors depend on sensor grade and result from well identified effects like : bias, scale factor, axes non orthogonality, noise, etc…

Earth gravity is not sensed by principle by accelerometers, hence it is accounted for by means of a mathematical model. Due to this modeling, any position error induces a gravity error which in turn results in a velocity error. Interaction arising between these position and velocity errors results in the Schuler oscillation which is known to affect long term inertial navigation.

The theory of navigation errors has been studied since the beginning of inertial navigation and is rather well documented in the bibliography (see for instance [1], [2], [3], [4]).

It is the feeling of the author however that this theory is not clearly assessed in most of articles because it is based on considering many complicated and artificial frames inherited from the time of gimbaled systems.

Now with strap down systems, the navigation solution is obtained by means of calculation only and, accordingly, it should be possible to investigate navigation errors by means of a purely analytic approach.

The purpose of present work is therefore to revisit the problem of navigation error theory with new eyes. To this end, navigation equations are considered and what happens when these equations are integrated with slightly wrong terms is examined by applying the perturbation method.

Equations of Navigation

Applications here considered concern terrestrial navigation i.e. locating a moving body with respect to the Earth. Body location is at point M on figure 1 and the projection of this point normally to Earth surface is represented by N. Earth shape is supposed to be ellipsoidal and Navigation frame used throughout the study is the so-called Local Level frame. Origin of this frame is at point N and its axes are respectively orientated to the North, East and Downward directions. Body location is defined by means of its geographic coordinates :

φ geodetic latitude

L longitude

h altitude above ellipsoidal Earth.

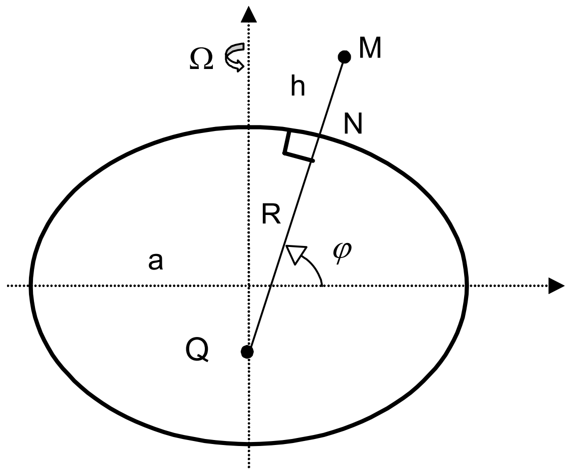

Fig. 1

Velocity of interest is made of the body displacement rate with respect to Earth, also referred as ground velocity. This velocity writes in the navigation frame as :

$$V^T = (V_N, V_E, V_D)$$

Equations processed in a strap down navigator are indicated below in (1), (2) and (3).

$$\dot{\varphi} = \frac{V_N}{R_m + h} \; ; \dot{L} = \frac{V_E}{(R+h)\cos\varphi} \; ; \dot{h} = -V_D \quad (1)$$

$$\dot{V} = [A]^T \Gamma_m + g - (\rho + 2\Omega) \wedge V \quad (2)$$

$$[\dot{A}] = -[\omega_m \times][A] + [A][(\rho + \Omega)\times] \quad (3)$$

Equations (1) directly come from body displacement kinematics and allow to determine the position of the body once ground velocity V_N, V_E and V_D is known.

Equation (2) states that the absolute acceleration applied to the IMU is equal to the sum of gravity and of the specific force sensed by the IMU accelerometers*. That equation is written in the navigation frame and accordingly absolute acceleration applied to the IMU is composed of Earth relative acceleration $\dot{V} + \rho \wedge V$, of Coriolis acceleration $2\Omega \wedge V$ and of centrifuge acceleration due to Earth rotation. ρ designates the transport rate ie the rotation rate of the navigation frame with respect to Earth while Ω represents the Earth rate.

Γ_m indicates the specific acceleration sensed by the strap down IMU. This vector is physically available through its components in body fixed frame so one has:

$$\Gamma_m{}^T = (\Gamma_{mX}, \Gamma_{mY}, \Gamma_{mZ})$$

Gravity g is the so-called "plumb-bob" gravity and stands both for the attraction force exerted by Earth mass and for the centrifuge acceleration due to Earth rotation. The theoretical model used to represent "plumb-bob" gravity is the so-called normal gravity model, given in the navigation frame by:

* Velocity obtained by integrating equation (2) represents the velocity of the body point located at the IMU center of measure whereas velocity mentioned in equation (1) is the velocity of body mass center. Velocity obtained from (2) has therefore to be corrected in navigation processing, in order to account for the distance existing between body overall mass center and IMU measurement. This lever arm correction however has a second order influence in error dynamics and is disregarded in present paper.

$$g^T = (0, 0, g)$$

With $$g = g_0(\varphi)\frac{a^2}{(a+h)^2} \quad (4)$$

a represents the Earth equatorial radius; R is the great normal in the meridian plane of the ellipsoid (distance NQ on Figure 1); R_m is the radius of curvature at point N in the meridian plane; $g_0(\varphi)$ expresses the tiny variation of gravity magnitude against latitude which results from Earth flattening.

Attitude matrix $[A]$ accounts for the angular position taken by the body with respect to the Navigation frame. Observe that this matrix is here defined in such a way as it brings the navigation frame in coincidence with the body fixed frame, and not the inverse.

Equation (3) deals with angular kinematics and relates the time variation of attitude matrix with the angular rate $\omega_m - (\rho + \Omega)$ experienced by the body relatively to the Navigation frame. ω_m represents the body rate sensed by the strap down IMU and is therefore available in the body frame as :

$$\omega_m^T = (\omega_{mX},\ \omega_{mY},\ \omega_{mZ})$$

Matrix $[\omega_m \times]$ is skew symmetric and is constructed from the components of ω_m as indicated below :

$$[\omega_m \times] = \begin{pmatrix} 0 & -\omega_{mZ} & \omega_{mY} \\ \omega_{mZ} & 0 & -\omega_{mX} \\ -\omega_{mY} & \omega_{mX} & 0 \end{pmatrix}$$

The transport rate ρ is expressed in the navigation frame as :

$$\rho^T = (\dot{L}\cos\varphi, -\dot{\varphi}, -\dot{L}\sin\varphi) \quad (5)$$

and the Earth rate as :

$$\Omega^T = (\Omega_0 \cos\varphi, 0, -\Omega_0 \sin\varphi) \quad (6)$$

Vectors ω_m and $\rho + \Omega$ being available in two different frames involve that skew symmetric matrices $[\omega_m \times]$ and $[(\rho + \Omega)\times]$ are not factored identically in right member of (3).

Variables defining navigation error

Navigation errors concerning either position, velocity or attitude are all assumed throughout the study to be of small amplitude. Accordingly, the analysis is restricted to first order developments.

Let us consider the position vector X going from the centre of the Earth to the IMU born by the moving body. **Position error** is made of the variation δX of this vector and is defined in the navigation frame as :

$$\delta X^T = (\delta X_N, \delta X_E, \delta X_D)$$

It is easily realized that this position error depends on geographic errors by means of :

$$\begin{aligned} \delta X_N &= (R_m + h)\,\delta\varphi \\ \delta X_E &= (R + h)\cos\varphi\,\delta L \\ \delta X_D &= -\,\delta h \end{aligned} \quad (7)$$

Velocity error is made of the errors which affect each component of ground velocity. Hence velocity error is represented by :

$$\delta V^T = (\delta V_N, \delta V_E, \delta V_D)$$

Attitude error is defined by considering the differential of each element of matrix $[A]$ and noting $\delta[A]$ the matrix made of these differentials. Within small angle assumption, it is found that, whatever the type of parameters chosen to represent attitude (Euler angles, quaternion), $\delta[A]$ can always be put, to first order of approximation, under the form below :

$$\delta[A] \cong -[A][\Phi\times] \qquad (8)$$

with $[\Phi\times]$ a skew symmetric matrix such as :

$$[\Phi\times] = \begin{pmatrix} 0 & -\Phi_D & \Phi_E \\ \Phi_D & 0 & -\Phi_N \\ -\Phi_E & \Phi_N & 0 \end{pmatrix}$$

Φ_N represents the body attitude error around the North direction and similarly Φ_E and Φ_D represent attitude errors around the East and Down directions. These three angles are grouped to define the vector of attitude error:

$$\Phi^T = (\Phi_N, \Phi_E, \Phi_D)$$

Since the Local Level frame has its origin bound to body location, it is easy to understand that any error occurring either in latitude or in longitude automatically results in a wrong orientation of the Navigation frame with respect to Earth. This error, called the *Orientation error*, is composed of small rotations about the North, East and Downward directions and is represented in vector notation as :

$$\delta\theta^T = (\delta\theta_N, \delta\theta_E, \delta\theta_D)$$

By analogy with (5), the components of $\delta\theta$ are given by :

$$\begin{aligned} \delta\theta_N &= \cos\varphi\,\delta L \\ \delta\theta_E &= -\delta\varphi \\ \delta\theta_D &= -\sin\varphi\,\delta L \end{aligned} \qquad (9)$$

Accounting for (7), it should be observed that $\delta\theta_N$, $\delta\theta_E$ and $\delta\theta_D$ are related with horizontal errors δX_N and δX_E. Therefore they do not constitute independent variables.

Dynamics of navigation error

At this point, it has been seen that variables associated with navigation errors are made of the 9 scalar quantities contained in vectors δX, δV and Φ. This section is now devoted to assess the relationships which relate these variables with the sensor measurement errors $\delta\Gamma_m$, $\delta\omega_m$ and with the gravity error δg.

Position error dynamics is essentially obtained by deriving (7) and differentiating (1). After some manipulation and accounting for (9), the following result holds :

$$\delta\dot{X} = \delta V - \rho\wedge\delta X + \delta\theta\wedge V \qquad (10)$$

$\delta\dot{X}$ represents the time variation of position error in the Navigation frame. Hence $\delta\dot{X} + \rho\wedge\delta X$ represents the time variation of the position error when this error is observed from an Earth fixed frame.

Equation (10) then simply means that the variation of the position error observed from an Earth fixed frame comes from the velocity error δV directly manifested in the navigation frame and from the projection error $\delta\theta\wedge V$ due to the slightly wrong orientation of the Navigation frame.

Velocity error dynamics is obtained by differentiating (2) and using (8) for expressing $\delta[A]^T$. This yields the equation below :

$$\begin{aligned}\delta\dot{V} = {} & \Phi\wedge f + \varepsilon_a + \delta g - (\rho + 2\Omega)\wedge\delta V - \\ & -(\delta\rho + 2\,\delta\Omega)\wedge V_R\end{aligned} \qquad (11)$$

with $f = [A]^T\Gamma_m$ and $\varepsilon_a = [A]^T\delta\Gamma_m$

f represents the specific acceleration measured by the IMU, projected onto the navigation frame. Similarly ε_a represents the IMU acceleration measurement error once projected into the navigation frame.

δg is obtained by differentiation of (4) and writes :

$$\delta g^T = (0\,,0\,,-\frac{2g}{a+h}\delta h) \qquad (12)$$

Attitude error dynamics is assessed by differentiating (3) and deriving (8). After some manipulation and transforming the resulting matrix equation into a vector equation, the following relation holds :

$$\dot{\Phi} = -(\rho + \Omega) \wedge \Phi + \varepsilon_g - (\delta\rho + \delta\Omega) \quad (13)$$

$\varepsilon_g = [A]^T \delta\omega_m$ represents the measurement error made by the IMU gyrometers.

At this point, it has been shown that navigation errors δX, δV and Φ satisfy the system of equations represented by relations (10), (11) and (13).

These equations are strictly identical with those mentioned in [2], [3] and correspond to what is called in bibliography the **"true frame"** approach or the **"phi angle"** approach.

One can notice however that equations (11) and (13) exhibit differential terms $\delta\rho$ and $\delta\Omega$ that must be resolved in terms of the problem variables if we were to obtain a closed system.

An essential equation related with **Orientation error dynamics** is obtained by deriving $\delta\theta$ against time. Considering (9) and (5) that equation writes as :

$$\delta\dot{\theta} = \delta\rho - \rho \wedge \delta\theta \quad (14)$$

At last, computing $\delta\Omega$ from (6) yields :

$$\delta\Omega = \Omega \wedge \delta\theta \quad (15)$$

Using (14) and (15), it is seen that $\delta\rho$ and $\delta\Omega$ could be eliminated in (11), (13) so that error equations constitute a closed system. However this system is poorly conditioned since (11) now depends on both $\delta\dot{V}$ and $\delta\dot{X}$ while (13) depends on $\dot{\Phi}$ and $\delta\dot{X}$.

Instead of trying to separate these intricate terms, a better solution is examined in the next section : it consists in modifying the problem variables in order to obtain simpler and better conditioned error equations.

Transformation of Navigation error equations

Let us consider the new variables indicated below for velocity and attitude errors :

$$\delta V_1 = \delta V + \delta\theta \wedge V \quad (16)$$
$$\Psi = \Phi + \delta\theta \quad (17)$$

Ψ is directly equal to the sum of the real attitude error Φ and of the orientation error $\delta\theta$. On its side, δV_1 represents the sum of the real velocity error δV and of the velocity error created by the wrong orientation of the navigation frame.

Using (16) and (17) it is a matter of calculation to show that error dynamics (10), (11), (13) is transformed into the system below :

$$\delta\dot{X} = \delta V_1 - \rho \wedge \delta X \quad (18)$$
$$\delta\dot{V}_1 = \Psi \wedge f + \varepsilon_a + \delta g_1 - (\rho + 2\Omega) \wedge \delta V_1 \quad (19)$$
$$\dot{\Psi} = -(\rho + \Omega) \wedge \Psi + \varepsilon_g \quad (20)$$

with : $\delta g_1 = \delta g + \delta\theta \wedge g$

Accounting for (12) and expressing $\delta\theta$ as a function of δX the gravity error is now defined as :

$$\delta g_1 = \frac{g}{a+h}(-\delta X_N, -\delta X_E, 2\,\delta X_D) \quad (21)$$

Navigation error dynamics (18) to (20) correspond to the so-called **"computer frame"** approach or **"psi angle"** approach and are identical with equations mentioned for instance in [2], [3]. This system of equations is well conditioned for integration and allows to predict the evolution of navigation error with good reliability.

The magnitude of orientation error is generally weak since $\delta\theta$ reaches only 0.15 mrad for a location error equal to 1 Km. Similarly $\delta\theta \wedge V$ represents only 0.05 m/s in the case of a vehicle moving at Mach 0.8 and having a location error of 1 Km. These figures indicate that Ψ only slightly departs from the true attitude error and that δV_1 remains very close to the true velocity error.

Reducing (18) and (19) to their major terms these equations write : $\delta \dot{X} \approx \delta V_1$ and $\delta \dot{V}_1 \approx \delta g_1$. Investigating the dynamic behavior of this linear system, it is found that horizontal errors δX_N, δV_{1N} (and similarly $\delta X_E, \delta V_{1E}$) oscillate at a period $T_S = 2\pi\sqrt{\frac{a+h}{g}}$ (Schuler period of 84 minutes) whereas vertical errors δX_D, δV_{1D} are unstable with an exponential time constant equal to $T_S / 2\pi\sqrt{2}$.

Finally, another change of variables leading to the so-called "**modified psi angle"** model [4] is worth to be noticed. That transformation consists in defining the velocity error as:

$$\delta V_2 = \delta V_1 - \Psi \wedge V \qquad (22)$$

Accounting for this new variable, relations (18) and (19) become:

$$\delta \dot{X} = \delta V_2 - \rho \wedge \delta X + \Psi \wedge V \qquad (23)$$

$$\delta \dot{V}_2 \cong \varepsilon_a - \varepsilon_g \wedge V + \delta g_2 - (\rho + 2\Omega) \wedge \delta V_2 \qquad (24)$$

with $\delta g_2 = \delta g_1 - \Psi \wedge g$.

The coefficients which intervene in right members of equation (24) no longer include the specific acceleration f but only the velocity V. This vanishing of f is favorable for computation since it is now possible to integrate error dynamics with a larger time step than previously with equation (19).

Finally note that this new change of variable also gives the formulae below:

$$\delta V_2 = \delta V - \Phi \wedge V$$
$$\delta g_2 = \delta g - \Phi \wedge g$$

Conclusion

Navigation error dynamics has been formulated in this work using a purely analytical approach, in a way coherent with the fact that navigation solution is totally derived by numerical processing in present day strap down initial systems.

Linear equations relating position error, velocity error, attitude error and IMU measurement errors have been assessed and are presented under three models knows as :

- the "true frame" model or "phi angle" model,
- the "computer frame" model or "psi angle" model,
- the "modified psi angle" model.

Influence of wrong orientation of the navigation frame induced by horizontal errors has been brought into light and finally Schuler oscillation and vertical channel divergence have been physically explained.

For navigation update, the "computer frame" model should be preferred although it doesn't directly involve the physical navigation errors.

The study was carried out for terrestrial navigation, however error dynamics could be transformed without great difficulty to cope for example with orbital navigation or with space navigation.

References

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